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Locally Coalescent Classes of Lie Algebras

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Introduction

In the recent study of infinite-dimensional Lie algebras, an important role has been played by the notion of coalescent classes of Lie algebras introduced in [5, 6]. A more general notion, locally coalescent classes, has been introduced by B. Hartley [5] and investigated by R. K. Amayo [2, 3, 4]. In this paper we shall develop a number of characterizations of locally coalescent classes of Lie algebras and investigate the radicals of Lie algebras defined by such classes.

In Section 2, we shall show several lemmas on locally coalescent and persistent classes for the subsequent sections. In Section 3, we shall show characterizations of locally coalescent classes by making use of the closure operations M and N. Namely, we show that a class $\mathfrak{X}$ is locally coalescent if and only if any class $\mathfrak{Y}$ such that $\mathfrak{X} \leq \mathfrak{Y} \leq \mathrm{M}\mathfrak{X}$ is locally coalescent and that, when the basic field is of characteristic 0 and $\mathfrak{X}$ is I-closed, $\mathfrak{X}$ is locally coalescent if and only if $\mathrm{M}\mathfrak{X} = \mathrm{N}\mathfrak{X}$, and if and only if $\mathrm{M}\mathfrak{X}$ is persistent (Theorems 3.2, 3.3 and 3.5). We also show that, if the basic field is of characteristic 0, $\mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$ is locally coalescent and persistent (Proposition 3.7).

In [3] it is stated that, when the basic field is of characteristic 0, B. Hartley has introduced the radical $\beta^*(L)$ of a Lie algebra L as the subalgebra generated by all the $\mathrm{L}\mathfrak{N}$ subideals of L, where $\mathrm{L}\mathfrak{N}$ is locally coalescent. More generally, we define the radical $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ of L for any locally coalescent class $\mathfrak{X}$ in a similar way and investigate its properties in Sections 4 and 5. We show that if $\mathfrak{X}$ is complete (resp. strongly complete), then $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ is invariant under every locally finite derivation (resp. every derivation) of L (Theorems 5.1 and 5.5). We also treat of the special case where $\mathfrak{X}$ is $\mathfrak{S} \cap \mathfrak{F}$, $\mathfrak{N}$ or $\mathfrak{S}$ (Corollaries 5.3 and 5.6).

§1. Preliminaries

We shall be concerned with Lie algebras over a field Φ which are not necessarily finite-dimensional. Throughout this paper, L will be an arbitrary Lie algebra over a field Φ and $\mathfrak{X}$ will be an arbitrary class of Lie algebras over Φ, that is, an arbitrary collection of Lie algebras over Φ such that $(0) \in \mathfrak{X}$ and if $H \in \mathfrak{X}$ and $H \simeq K$ then $K \in \mathfrak{X}$, unless otherwise specified.