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Non-triviality of an Element in the Stable Homotopy Groups of Spheres

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Statement of results

In the stable homotopy groups G_* of spheres, two non-trivial families of p-primary elements, called α- and β-series, are known [6] (cf. [12]). These are constructed from the attaching classes α and β of the spectra $V(1)$ and $V(2)$ [12], whose cohomology groups are certain exterior algebras over the Steenrod algebra $\bmod p$ [11]. In a similar way, the existence of the spectrum $V\left(2\frac{1}{2}\right)$ assures to define an element called γ_1[12; § 5], which is the first element of the third family.

The purpose of this paper is to prove the following result.

MAIN THEOREM. *For every prime* $p \geqq 5$, *the element* $\gamma_1 \in G_{(p^2-1)q-3}$, $q = 2(p-1)$, *is non-trivial.*

The result is an answer to a problem proposed by one of the authors [12; p. 237], and P. E. Thomas and R. Zahler [7] [13] also have obtained the same result in a quite different method. Our result states that γ_1 is a non-zero multiple of the element $\alpha_1\beta_{p-1}$ [12; (5.12)]. Also, one of the authors recently has proved more strict relation $\gamma_1 = \alpha_1\beta_{p-1}$.

Originally, this paper was intended to prove $\gamma_1 = 0$ (cf. [4; II, Remark in p. 147], [7; § 0]), but the publication has been postponed by a contradiction to the result of P. E. Thomas and R. Zahler. We have re-examined our original proof, and after crucial investigations we have concluded the opposite result.

COROLLARY 1 ($p \geqq 5$). *The following relations hold in* G_*:

$$\alpha_1\beta_{p-1}\beta_s = 0 \qquad \textit{for} \quad s \geqq 3,$$

and hence

$$\alpha_1\beta_1\beta_k = \alpha_1\beta_2\beta_{k-1} = 0 \qquad \textit{for} \quad k \not\equiv -2 \bmod p \quad \textit{and} \quad k \geqq p+1,$$

$$\alpha_1\beta_1^2\beta_k = \alpha_1\beta_1\beta_2\beta_{k-1} = 0 \qquad \textit{for} \quad k \geqq p+1.$$

This is an easy restatement of Proposition 5.9 of [12]. Also, by Corollary 5.7, Theorem 5.1 and (5.4) of [12], we obtain the parallel relations in the algebra