HIROSHIMA MATH. J.
5 (1975), 87–114

A New Family in the Stable Homotopy Groups of Spheres

Shichirô OKA

(Received September 9, 1974)

Introduction

Let G_k denote the k-th stable homotopy group Dir $\lim \pi_{N+k}(S^N)$ of spheres. J. F. Adams [0] and H. Toda [9] discovered a family $\{\alpha_t \in G_{tq-1}, t \geqq 1\}$, $q = 2(p-1)$, of elements of order p, for every odd prime p, and later on L. Smith [6] and H. Toda [11] discovered another family $\{\beta_t \in G_{(tp+t-1)q-2}, t \geqq 1\}$ of elements of order p, for every prime $p \geqq 5$. Our main results concern the second family.

THEOREM A. *For every prime $p \geqq 5$ and $t \geqq 1$, there exist $p-1$ elements*

$$\rho_{t,r} \in G_{(tp^2+(t-1)p+r)q-2}, \qquad r = 1, 2, \cdots, p-1,$$

of order p such that

$$\rho_{t,r+s} \in \langle \rho_{t,r}, p, \alpha_s \rangle \qquad for \quad r+s \leqq p-1$$

and that the last element $\rho_{t,p-1}$ coincides with the element β_{tp} of L. Smith [6] *and H. Toda* [11]. *Here, $q=2(p-1)$ and $\langle\ ,\ ,\ \rangle$ denotes the stable Toda bracket.*

For $t=1$, this family $\{\rho_{1,r}\}$ coincides with the family $\{\varepsilon_r \in G_{(p^2+r)q-2}, 1 \leqq r \leqq p-1\}$ constructed in [3].

Let M be a Moore space $S^1 \cup_p e^2$ and denote by $\mathscr{A}_k(M)$ the limit group Dir $\lim [S^{N+k}M, S^N M]$. Let $i: S^1 \to M$ and $\pi: M \to S^2$ be the natural maps and consider the induced homomorphism $\pi_* i^*: \mathscr{A}_k(M) \to G_{k-1}$.

There exists uniquely an element $\alpha \in \mathscr{A}_q(M)$, $q=2(p-1)$, such that $\pi_* i^* \alpha = \alpha_1$, and also there exists a family $\{\beta_{(t)} \in \mathscr{A}_{(tp+t-1)q-1}, t \geqq 1\}$ of $\mathscr{A}_*(M)$ which satisfies $\alpha\beta_{(t)} = \beta_{(t)}\alpha = 0$ and $\beta_{(t)} \in \langle \beta_{(t-1)}, \alpha, \beta_{(1)} \rangle$ [11] (cf. [4]). This family is closely connected with the family $\{\beta_t\}$ via the equality $\pi_* i^* \beta_{(t)} = \beta_t$, and our next results are related to the α-*divisibility* of the elements $\beta_{(tp)}$, $t \geqq 1$.

We constructed in [4] the element ε of $\mathscr{A}_{(p^2+1)q-1}(M)$, which is a generator of the ring $\mathscr{A}_*(M)$. The element $\pi_* i^* \varepsilon$ generates the p-component of $G_{(p^2+1)q-2}$ and there is a relation $\varepsilon\alpha^{p-2} = \alpha^{p-2}\varepsilon = \beta_{(p)}$. Also we defined in [4] a differential D on $\mathscr{A}_*(M)$ of degree $+1$, originally due to P. Hoffman. D is a derivation and the subring $\mathrm{Ker}\, D$ is commutative in graded sense. Our elements α, $\beta_{(t)}$ and ε