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On the Group of Self-Equivalences of the Product of Spheres

Norichika SAWASHITA

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§1. Introduction

The set $\mathscr{E}(X)$ of homotopy classes of self-(homotopy-)equivalences of a based space X forms a group by the composition of maps, and this group is studied by several authors.

The purpose of this note is to study the groups $\mathscr{E}(S^m \times S^n)$ of the products $S^m \times S^n$, where S^k is the k-sphere. These are studied by P. J. Kahn [8] for the case $m=n$, and by A. J. Sieradski [13] for the case $m, n=1, 3, 7$.

In the first, we consider the case $n>m\geqq 2$. Then the wedge $S^m \vee S^n$ is simply connected, and we can apply the results of [10, §§1–2] to the mapping cone $S^m \times S^n = (S^m \vee S^n) \cup e^{m+n}$ of the Whitehead product. Hence, by using the results of W. D. Barcus and M. G. Barratt [3, §4], we have in Theorem 2.6 the exact sequence

$$0 \longrightarrow H_{m,n} \longrightarrow \mathscr{E}(S^m \times S^n) \longrightarrow G_{m,n} \longrightarrow 1,$$

where $H_{m,n}$ is the factor group of $\pi_{m+n}(S^m)+\pi_{m+n}(S^n)$ and $G_{m,n}$ is the subgroup of $\mathscr{E}(S^m \vee S^n)$. In §3, we study some cases that this sequence is split, but the extension of this sequence is not known to us in general. Also, by using the quaternion, we compute $\mathscr{E}(S^m \times S^n)$ for $m=2, 3$ and $n>m$ in Theorems 4.3 and 5.3, and we see that the above sequence is split if $m=2$ and is not split if $m=3$ and $n=5$.

By the same way, we have in Theorem 6.2 the similar exact sequence for the case $n=m\geqq 2$, which is split if n is even. Furthermore, we can determine the group $\mathscr{E}(S^n \times S^n)$ for $n=3, 7$ in Theorem 6.4.

The group $\mathscr{E}(S^1 \times S^n)$ is computed in §§7–8 by the different methods. By attaching i-cells ($i\geqq n+3$) to S^n, we obtain a CW-complex X_{n+1} which kills the r-th homotopy groups of S^n for $r\geqq n+2$, and we see that $\mathscr{E}(S^1 \times S^n)$ is isomorphic to $\mathscr{E}(S^1 \times X_{n+1})$ (Lemma 7.1). Consider the composition

$$f\colon S^1 \times K(Z, n) \longrightarrow K(Z, n) \xrightarrow{f'} K(\pi_{n+1}(S^n), n+2)$$

of the natural projection and the generator f' of $H^{n+2}(Z, n; \pi_{n+1}(S^n))$. Then, it is well known that $S^1 \times X_{n+1}$ is the mapping track E_f of f. Hence, we can apply the results of J. W. Rutter [11] and [10, §5] to $\mathscr{E}(S^1 \times X_{n+1})$, and the