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On Conformal Invariants of Higher Order

Toshio NASU

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Let (M, g) be an n-dimensional Riemannian manifold with fundamental metric tensor g $(n>2)$ and R be the curvature tensor of type $(0, 4)$. Let C and C_0 be the Weyl conformal curvature tensor of type $(0, 4)$ and the so-called Weyl 3-index tensor, respectively. As usual, a Riemannian manifold is said to be *flat* or *of constant curvature* according as the sectional curvature is identically zero or constant, and to be *conformally flat* if it is locally conformally diffeomorphic to a Euclidean space. A well-known theorem due to H. Weyl says that (M, g) is conformally flat if and only if $C=0$ for $n>3$ and $C_0=0$ for $n=3$. The tensors R and C are typical examples of curvature structures of order two.

On the other hand, researches on curvature structures of higher order, e.g. the q-th Gauss-Kronecker curvature tensor R^q, have been developed by many people. Especially, J. A. Thorpe [7] has considered the $2q$-th sectional curvature γ_{2q}, which is defined for each even positive integer $2q \leqq n$, and studied relationships between curvature properties and topological structures of the manifold. The sectional curvature γ_{2q} is a curvature function corresponding to R^q on the Grassmann bundle of $2q$-planes tangent to the manifold, and coincides with the usual sectional curvature if $q=1$. The higher order sectional curvatures are weaker invariants of Riemannian structure than the usual sectional curvature.

Very recently, R. S. Kulkarni [4] has introduced an interesting double form $\operatorname{con}\omega$ for a double form ω, such as $\operatorname{con}R=C$ as a special case $\omega=R$. He also proved that $\operatorname{con}\omega$ has the same algebraic properties as the tensor C. It seems natural to seek for generalizations of classical results (conformal invariants, the theorem of Weyl etc.) on a conformal change of metric to the case of higher order, by making use of the Gauss-Kronecker curvature tensors. This is the purpose of the present work.

Section 1 is devoted to preliminary remarks. We shall recall definitions and fundamental formulas related to curvature structures from a view-point of double forms. In Section 2, we shall define a double form $\operatorname{con}_0\omega$ as a generalization of the Weyl 3-index tensor C_0 and obtain a new differential identity in Proposition 1. In Proposition 2, we shall give the conformal transformation formulas of $\operatorname{con} R^q$ and $\operatorname{con}_0 R^q$.

In this paper, a Riemannian manifold is said to be *q-flat* or *of q-constant curvature* according as the $2q$-th sectional curvature γ_{2q} is identically zero or constant, and to be *q-conformally flat* if $\operatorname{con} R^q=0$ for $n>4q-1$ and $\operatorname{con}_0 R^q=0$