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Hyperpolynomial Approximation of Solutions of Hereditary Systems

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1. Introduction

Consider an operator L on $C[0, \tau]$, where $C[0, \tau]=\{\phi | \phi: [0, \tau] \rightarrow R^n$, continuous$\}$ with norm $\|\cdot\|$. Suppose that x is a solution of the equation $L(x)=h$, subject to the initial condition $x(0)=\alpha$. Then a problem in approximation theory is whether there are hyperpolynomials $S_n^* \in \Pi_n^*$ (Π_n^* is the set of all hyperpolynomials S_n^* of degree less than or equal to n, which satisfy the condition $S_n^*(0)=\alpha$, [5]) such that $\|L(x)-L(S_n^*)\| = \inf_{S \in \Pi_n^*} \|L(x)-L(S)\|$, $n=1, 2, \dots$ and $\lim_{n \to \infty} S_n^* = x$, uniformly on $[0, \tau]$.

The above problem has been studied in the following cases:

i) $L(x) \equiv x' + B(t, x)$, $\|\cdot\| = \|\cdot\|_p$ (L_p-norm), $1 \leqq p \leqq \infty$. ([1], [3], [4].)

ii) $L(x) \equiv x' + B(t, x) + \int_0^t F(t, s, x(s)) ds$, $\|\cdot\| = \|\cdot\|_p$, $1 < p \leqq \infty$. ([5].)

The purpose of this paper is to study the same problem when L is an operator, which gives a hereditary system [2] and $\|\cdot\| = \|\cdot\|_p$, $1 \leqq p \leqq \infty$. The results here generalize those of [1], [3], [4], [5] not only for the case of the L_p-norm, $1 < p \leqq \infty$ but also for the L_1-norm.

2. Preliminaries

Let I be an interval of R, $A \subseteqq R$ be compact with $\max A = 0$, $\alpha: I \times A \rightarrow R$ be a continuous function, nondecreasing with respect to the second variable and $\alpha(t, 0)=t$, $t \in I$. If $x: \alpha(I, A) \rightarrow R^n$ is continuous and $C(A)=\{f | f: A \rightarrow R^n$, continuous$\}$, we define an operator $Q_t x: I \rightarrow C(A)$ by the relation

$$(Q_t x)(\theta) = x(\alpha(t, \theta)), \qquad t \in I, \quad \theta \in A.$$

An hereditary differential system is a relation of the form

$$(x - g(t, Q_t x))' = f(t, Q_t x)$$

where $f, g: I \times C(A) \rightarrow R^n$ are continuous.

Suppose $U \subseteqq C(A)$ is open. We say that a continuous function $g: U \rightarrow R^n$