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A Note on Finite Groups which Act Freely on Closed Surfaces

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§1. Introduction

The purpose of this note is to study what kind of finite groups can act freely on closed surfaces.

Let X be a given closed surface. Suppose that a finite group G acts freely on X. Then, it is well known that the orbit space $Y=X/G$ is also a closed surface and there is a normal covering

$$p: X \longrightarrow Y = X/G, \tag{1.1}$$

that is, the image $p_*\pi_1(X)$ of the induced monomorphism $p_*: \pi_1(X) \to \pi_1(Y)$ of the fundamental groups is a normal subgroup of $\pi_1(Y)$ and $\pi_1(Y)/p_*\pi_1(X) \cong G$. Therefore,

$$\chi(X) = \chi(Y)g \qquad (g \geqq 1), \tag{1.2}$$

where χ means the Euler characteristic and $g = \# G$ is the order of G. Also, we see easily the following.

(1.3) In the case that X is orientable, Y is orientable if and only if the action of G preserves the orientation of X.

Conversely, suppose that

(1.4.1) Y is a closed surface satisfying (1.2) for some integer $g \geqq 1$, and N is a normal subgroup of $\pi_1(Y)$ of index g, and

(1.4.2) N is isomorphic to $\pi_1(X)$.

Then, we have a normal covering $p': X' \to Y$ with the covering group

$$G = \pi_1(Y)/N, \tag{1.5}$$

and the closed surface X' satisfies $\chi(X') = \chi(X)$, $\pi_1(X') \cong \pi_1(X)$ by (1.4.1–2). Therefore, we see that X' is homeomorphic to X by the classification theorem of closed surfaces, and so G acts freely on X.

Thus, we have the following

THEOREM 1.6. *Let X be a closed surface. Then, a finite group G acts freely on X if and only if G is given by* (1.5) *under the assumptions* (1.4.1–2).

Furthermore, in the case that X is orientable, G acts on X preserving or