HIROSHIMA MATH. J.
5 (1975), 197-214

Dirichlet Integral of Product of Functions on a Self-adjoint Harmonic Space

Fumi-Yuki MAEDA

(Received January 16, 1975)

Introduction

In the previous paper [2], the author defined a notion of gradient measures for functions on a self-adjoint harmonic space. In case the harmonic space is given by solutions of a second order elliptic partial differential equation of the form

$$\sum_{i,j=1}^{k} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)-qu=0$$

on a Euclidean domain, the mutual gradient measure $\delta_{[f,g]}$ of functions f and g is given by

$$\delta_{[f,g]}=\left(\sum_{i,j=1}^{k} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\right)dx \qquad (dx\text{: the Lebesgue measure}).$$

Thus, in this case, the equality

$$(*) \qquad \delta_{[fg,\phi]}=f\delta_{[g,\phi]}+g\delta_{[f,\phi]}$$

holds. The main purpose of this paper is to show that the equality (*) remains valid for general self-adjoint harmonic spaces. Once this equality is established, we can consider Royden's algebra (cf. [3, Chap. III]) on a self-adjoint harmonic space. We shall also see that if the harmonic structure is considered on a Euclidean domain and satisfies a certain additional condition (see Theorem 5), then the gradient measure is expressed as

$$\delta_{[f,g]}=\sum_{i,j=1}^{k}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\nu_{ij}$$

with a positive-definite system of signed measures (ν_{ij}); and the harmonic functions are "solutions" of the second order elliptic partial differential equation

$$\sum_{i,j=1}^{k}\frac{\partial}{\partial x_i}\left(\nu_{ij}\frac{\partial u}{\partial x_j}\right)-\pi u=0$$

whose coefficients ν_{ij}, π are (signed) measures.