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The Stable Homotopy Groups of Spheres III

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Introduction

The present paper is the third part of the series [8]. §§1–8 are contained in Part I, §§9–17 are contained in Part II, and this part consists of §§18–23. We shall use all notations and notions defined in the previous parts.

Throughout this paper, we shall always assume that the fixed prime p is greater than 3, and not treat the 3-primary component. Our results in the previous parts are summarized in [8–II; Th. A], where we determined the group ${}_p\pi_k(S)$, the p-primary component of the k-th stable homotopy group of spheres, for $k \leqq (p^2+3p+1)q-6$, $q=2(p-1)$, and the first unsolved problem was to determine the composition $\alpha_1\beta_1\beta_{p+1}$. In [10] we have obtained the relations $\alpha_1\beta_1\beta_{p+s}=0$, $1 \leqq s \leqq p-3$, which enable us to extend our calculations.

In this part, we shall determine the group ${}_p\pi_k(S)$ for $(p^2+3p+1)q-5 \leqq k \leqq (2p^2+p)q-4$. In this range, there appear the following new generators:

$$\kappa_s = \{\beta_1\beta_{p+s}, \alpha_1, \alpha_1\}^{**)} \quad \textit{of degree} \quad (p^2+(s+2)p+s+1)q-5,\ 1 \leqq s \leqq p-3,$$

$$\lambda' = \{\beta_1^p, \varepsilon', \alpha_1\} \quad \textit{of degree} \quad (2p^2+1)q-5,$$

$$\lambda_1 = \{\varepsilon_1, \beta_1^p, \alpha_1\} \quad \textit{of degree} \quad (2p^2+1)q-4,$$

$$\lambda_i = \{\lambda_{i-1}, p\iota, \alpha_1\} \quad \textit{of degree} \quad (2p^2+i)q-4, \quad 2 \leqq i \leqq p-3,$$

$$\mu \in \{\lambda_{p-3} - y\beta_1\beta_{2p-2}, \alpha_1, \alpha_1\} \quad \textit{of degree} \quad (2p^2+p-1)q-5,$$

where $y \in Z_p$ is the coefficient in the relation $\alpha_1\lambda_{p-3} = y\alpha_1\beta_1\beta_{2p-2}$ and the orders of κ_s, λ', λ_i and μ are p, p, p and p^2, respectively. These elements together with the *α-families* $\{\alpha_r\}$, $\{\alpha'_{rp}\}$, $\{\alpha''_{rp^2}\}$ ([1], [9; §4], [14–IV]) and the *β-family* $\{\beta_r\}$ ([12], [17]) form a multiplicatively generating set for ${}_p\pi_k(S)$ in the cited range of k. Here, the orders of α_r, α'_{rp}, α''_{rp^2} and β_r are p, p^2, p^3 and p, respectively, and $\deg\alpha_r = rq-1$, $\deg\alpha'_{rp} = rpq-1$, $\deg\alpha''_{rp^2} = rp^2q-1$ and $\deg\beta_r = (rp+r-1)q-2$.

Our main results for ${}_p\pi_k(S)$ are Theorems 19.9, 21.6, 22.2 and 22.3, which are summarized in the following

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**) $\alpha = \{\beta, \gamma, \delta\}$ means that a secondary composition $\{\beta, \gamma, \delta\}$ consists of a single element α.