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On the Commutativity of Torsion and Injective Hull

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Introduction

Throughout this note A denotes a commutative ring with a unit and all modules are unitary A-modules. For any module M, if L is a submodule of M and S is a subset of M, then we put $(L: S)=\{x \in A; xS \subseteq L\}$, in particular $0(S)=(0: S)$. For any filter F of ideals of A, we have an operation upon the lattice of submodules of any A-module M, as follows. If L is a submodule of M, we define $C(L, M)=\{x \in M; (L: x) \in F\}$. Especially we rewrite $C(0, M)=T(M)$; $C(M, E(M))=D(M)$, where $E(M)$ is an injective hull of M. Our main purpose is to answer the question: With the above notations, let F' be another filter and T', D' be the associated operators relative to F'. Can we have the equalities
(1) $D'(T(M))=T(D'(M))$, (2) $D'(M/T(M))=D'(M)/D'(T(M))$ and
(3) $D(\mathrm{Hom}(N, M))=\mathrm{Hom}(N, D(M))$?

The above equalities have been obtained, in [8], in a special case using the local property.

§1. Notation and Preliminaries

Let F be a filter of ideals of A. When L is a submodule of an A-module M, we put $C(L, M)=\{x \in M; (L: x) \in F\}$. Especially we rewrite $C(0, M)=T(M)$, which is called the F-torsion of M; $C(M, E(M))=D(M)$; $C(\mathfrak{a}, A)=c(\mathfrak{a})$. It is easy to see that, for any submodule N of M, $C(L, M) \cap N=C(L \cap N, N)$ and $C(L, M)/L=T(M/L)$. We denote the class of A-modules M such that $T(M)=M$ by $\mathscr{T}$ and the class of A-modules M such that $T(M)=0$ by $\mathscr{F}$. The following facts are easy and well-known:

(1) The class $\mathscr{T}$ is closed under submodule, image and direct sum (such class will be called a weak torsion class). Hence a module M belongs to $\mathscr{T}$ if and only if $Ax \in \mathscr{T}$ for any element x in M.

(2) T is a left exact subfunctor. Namely, the functor T satisfies the properties: (i) $T(M) \subseteq M$, (ii) if L is a submodule of M, then $T(L)=T(M) \cap L$, and (iii) for any homomorphism $f: M \to N, f(T(M)) \subset T(N)$ (such functor is called a left exact preradical).

(3) The operator c satisfies the properties: (i) $\mathfrak{a} \subseteq c(\mathfrak{a})$, (ii) $c(\mathfrak{a} \cap \mathfrak{b})=c(\mathfrak{a}) \cap c(\mathfrak{b})$ and (iii) $(c(\mathfrak{a}): x)=c(\mathfrak{a}: x)$, for any ideals $\mathfrak{a}$, $\mathfrak{b}$ and any element x in A