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Notes on the Brauer Liftings of Finite Classical Groups

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Introduction

Let k be a finite field with algebraic closure K, let H be a finite group and let $r: H \to GL_m(k)$ be a (modular) representation of H. For $x \in H$, let $u_1(x),\ldots, u_m(x)$ denote the eigenvalues of $r(x)$. Define the complex valued function $b_{r,\theta}$ on H by $b_{r,\theta}(x) = \sum_{i=1}^{m} \theta(u_i(x))$, where θ is a character $K^\times \to \mathbf{C}^\times$. J. A. Green [5] proved that $b_{r,\theta}$ is then a generalized character of H, i.e. an integral linear combination of irreducible characters of H. In this paper we call $b_{r,\theta}$ the Brauer lifting of r associated to θ. It seems interesting to know the irreducible constituents of $b_{r,\theta}$ for a finite Chevalley group H, i.e. a finite group of k-rational points of a connected reductive linear algebraic group defined over k. For $H = GL_m(k)$, r the natural representation, J. A. Green [5] decomposed $b_{r,\theta}$, and when θ is in general position he obtained an important irreducible character, a cuspidal character.

We are interested in other classical groups $H = SO_{2n+1}(k)$, $GSp_{2n}(k),\ldots$ etc. Let r be the natural representation $H \to GL_m(k)$ and assume that θ is injective. If the number of elements in k is greater than 3, then the inner product on H, $< b_{r,\theta}, b_{r,\theta} >_H$ equals m. This is proved in § 2 by making use of a certain inner product formula, which is the simplest one among those obtained by N. Kawanaka [7]. Next in § 3, using an induction argument, we decompose $b_{r,\theta}$ into an alternating sum of irreducible characters. The same result is announced by G. Lusztig [10] at Vancouver Congress of I. C. M. and when $H = GL_m(k)$, T. A. Springer [12] has decomposed $b_{r,\theta}$ using the similar method to ours.

In the case of the group of symplectic similitudes $H = GSp_{2n}(k)$, we have the following result.

As maximal parabolic subgroups of $GSp_{2n}(k)$, we choose

$$P_o = \left\{ \begin{bmatrix} A & * \\ O & D \end{bmatrix} \in GSp_{2n}(k) \mid A,\ D \in GL_n(k) \right\},$$

$$P_i = \left\{ \begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix} \in GSp_{2n}(k) \mid A,\ D \in GL_{n-i}(k),\ X \in GSp_{2i}(k) \right\},\ (i=1,\ldots, n-1).$$