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Positive Bounded Solutions for a Class of Linear Delay Differential Equations

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Let n be an integer, $n \geq 2$, let q be a continuous function from $[0, \infty)$ to $(0, \infty)$, and let G be the set to which g belongs if and only if g is a nondecreasing unbounded continuous function from $[0, \infty)$ to $[0, \infty)$ such that $g(t) \leq t$ whenever $t \geq 0$. Let G° be that subset of G to which g belongs if and only if g is in G and $g(t) < t$ whenever $t > 0$. We propose to study the differential equation

$$u^{(n)}(t) + (-1)^{n+1} q(t) u(g(t)) = 0, \tag{1}$$

for g in G. A function u from $[0, \infty)$ to $(-\infty, \infty)$ is called a solution of (1) if and only if there is $b \geq 0$ such that $u^{(n)}$ exists on (b, ∞) and (1) is true whenever $t > b$. A solution u of (1) is called oscillatory if and only if the set $\{t : t \geq 0$ and $u(t) = 0\}$ is unbounded. Otherwise, u is called nonoscillatory. Although the analogue of (1) without delay is known to have a positive bounded solution, several authors have shown that if the delay is large enough, i.e., g is small enough, then every bounded solution of (1) is oscillatory. In particular, if g is in G, if

$$\int_0^\infty t^{n-1} q(t) dt = \infty, \tag{2}$$

and if

$$\limsup_{t \to \infty} \int_{g(t)}^t (g(t) - g(s))^{n-1} q(s) ds > (n-1)!, \tag{3}$$

then G. Ladas, V. Lakshmikantham, and J. S. Papadakis [3] have shown that every bounded solution of (1) is oscillatory. M. Naito [7] has shown that if g is in G and

$$\limsup_{t \to \infty} \int_{g(t)}^t (s - g(t))^{n-1} q(s) ds > (n-1)!, \tag{4}$$

then every bounded solution of (1) is oscillatory. Note that although each of (3) and (4) implies (2), (3) and (4) are independent. Since the results of [3] and [7] are of the nature "if g is small enough then every bounded solution of (1) is oscillatory", the question arises: If g is large enough can we conclude the existence of a positive bounded solution? We shall give a result which answers