HIROSHIMA MATH. J.
6 (1976), 421–428

Certain Functional of Probability Measures on Hilbert Spaces

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(Received February 9, 1976)

§1. Introduction and results

Let E be a real separable Hilbert space with inner product $<\ ,\ >$ and $\mathscr{E}$ the σ-algebra of all Borel subsets of E. We denote by $\mathscr{P}$ the set of all probability measures μ on $(E, \mathscr{E})$ with a finite second moment; $\int \|x\|^2 d\mu(x) < \infty$. For each $\mu \in \mathscr{P}$ there exist a vector m (mean vector) and a bounded linear operator V (covariance operator) with $\int <x, u> d\mu(x) = <m, u>$ and $\int <x-m, u> <x-m, v> d\mu(x) = <Vu, v>$ for all $u, v \in E$. Since the covariance operator is symmetric, non-negative and nuclear, we can find a unique Gaussian measure γ_μ on $(E, \mathscr{E})$ which has the same mean vector and covariance operator as those of μ [4; p. 14 and p. 18]. Let $\mathscr{M}(\mu)$ be the set of all probability measures M on $(E \times E, \mathscr{E} \otimes \mathscr{E})$ with $M(A \times E) = \mu(A)$ and $M(E \times A) = \gamma_\mu(A)$ for all $A \in \mathscr{E}$. We consider a function: $M \to e[\mu; M] = \iint \|x-y\|^2 dM(x, y)$ on $\mathscr{M}(\mu)$, and define a functional e on $\mathscr{P}$ by

$$e[\mu] = \inf_{M \in \mathscr{M}(\mu)} e[\mu; M].$$

The functional e was first introduced by H. Tanaka in the case where E is the one-dimensional space and its basic properties were studied also by himself [5]. H. Murata and H. Tanaka [2] extended the results to the case of multi-dimensional Euclidean spaces.

The purpose of this paper is to show that some of their results can be extended to the case of Hilbert spaces, by the method similar to that of [2] with a slight simplification. That is, we shall prove:

THEOREM 1. *For each $\mu \in \mathscr{P}$ there exists an $M \in \mathscr{M}(\mu)$ with $e[\mu] = e[\mu; M]$ and such a measure M has the form; $M(A \times B) = \gamma_\mu(f^{-1}(A) \cap B)$ for all $A, B \in \mathscr{E}$ with a Borel measurable mapping f from E into itself. Consequently $e[\mu] = \int \|f(y) - y\|^2 d\gamma_\mu(y)$.*

THEOREM 2. *Let μ_1 and μ_2 be measures in $\mathscr{P}$ and $\mu_1 * \mu_2$ their convolution. Then*