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Potential Theoretic Properties of the Subdifferential of a Convex Function

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1. Introduction

Throughout this paper, let X be a locally compact Hausdorff space and ξ be a positive (Radon) measure on X. We denote by $L^2 = L^2(X; \xi)$ the Hilbert space of all real-valued square ξ-integrable functions on X with the inner product $(u, v) = \int_X u \cdot v d\xi$ and the norm $\|u\| = \sqrt{(u, u)}$. For measurable functions u and v on X we define $u \vee v = \max\{u, v\}$, $u \wedge v = \min\{u, v\}$, $u^+ = u \vee 0$ and $u^- = -(u \wedge 0)$ and simply write "$u \leqq v$" for "$u \leqq v$ ξ-a.e. on X".

Let ϕ be a lower semicontinuous convex function on L^2 with values in $(-\infty, +\infty]$ and proper on L^2, i.e., $\phi \not\equiv +\infty$ on L^2. Then the subdifferential $\partial\phi$ of ϕ is the (multivalued) operator in L^2 defined by the following: $\partial\phi(u) = \emptyset$ if $u \notin D(\phi)$ and

$$\partial\phi(u) = \{w \in L^2; (w, v-u) \leqq \phi(v) - \phi(u) \quad \text{for all} \quad v \in L^2\}$$

if $u \in D(\phi)$, where $D(\phi) = \{v \in L^2; \phi(v) < \infty\}$. We put

$$D(\partial\phi) = \{u \in L^2; \partial\phi(u) \neq \emptyset\}$$

and

$$G(\partial\phi) = \{[u, v] \in L^2 \times L^2; u \in D(\partial\phi), v \in \partial\phi(u)\},$$

which are called the domain and the graph of $\partial\phi$, respectively.

In the Dirichlet space theory, contractions on the real line play an important role in connection with potential theoretic properties. Among them the following are the most fundamental:

(a) $T^+t = \max\{t, 0\}$ (*positive contraction*)

(b) $T_1^+t = \min\{T^+t, 1\}$ (*unit contraction*).

In case ϕ is Gâteaux-differentiable on a certain functional space, Kenmochi-Mizuta [6, 7] discussed relations between the above contractions and potential theoretic properties, e.g., the maximum principle, the principle of lower envelope, the complete maximum principle and the strong principle of lower envelope.