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On the Radial Limits of Potentials and Angular Limits of Harmonic Functions

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1. Introduction

In the n-dimensional Euclidean space R^n ($n \geqq 2$), the Riesz potential of order α of a non-negative function f in $L^p(R^n)$ is defined by

$$U_\alpha^f(x) = \int |x - y|^{\alpha-n} f(y) dy, \qquad x \in R^n,$$

where $0<\alpha<n$ and $1<p<\infty$. Our first aim is to discuss the existence of radial limits of U_α^f at a point of R^n, which can be assumed to be the origin O of R^n without loss of generality. For this purpose we shall use the capacity $C_{\alpha,p}$, which is a special case of the capacity $C_{k;\mu;p}$ introduced by N. G. Meyers [4] and is defined by

$$C_{\alpha,p}(E) = \inf \|g\|_p^p, \qquad E \subset R^n,$$

the infimum being taken over all non-negative functions $g \in L^p(R^n)$ such that $U_\alpha^g(x) \geqq 1$ for all $x \in E$; in case $\alpha p \geqq n$, we assume further that g vanishes outside the open ball with center at O and radius 2. In § 3, setting $S = \{x \in R^n; |x| = 1\}$, we shall show that for a non-negative function $f \in L^p(R^n)$ satisfying $\int |y|^{\alpha p - n} f(y)^p dy < \infty$,

$$\text{(i)} \qquad \lim_{r \downarrow 0} U_\alpha^f(r\xi) = U_\alpha^f(O)$$

holds for $\xi \in S$ except those in a Borel set with $C_{\alpha,p}$-capacity zero. In case $U_\alpha^f(O) = \infty$, $\lim_{x \to O} U_\alpha^f(x) = U_\alpha^f(O)$ by the lower semi-continuity of U_α^f, and hence (i) holds for all $\xi \in S$. In this case, we shall investigate the order of infinity; in fact, we shall show that if $\alpha p \leqq n$ and f is a non-negative function in $L^p(R^n)$ with $U_\alpha^f \not\equiv \infty$, then we have

$$\begin{cases} \lim_{r \downarrow 0} r^{(n-\alpha p)/p} U_\alpha^f(r\xi) = 0 & \text{in case } \alpha p < n, \\ \lim_{r \downarrow 0} \left(\log \frac{1}{r}\right)^{1/p-1} U_\alpha^f(r\xi) = 0 & \text{in case } \alpha p = n \end{cases}$$

for $\xi \in S$ except those in a Borel set with $C_{\alpha,p}$-capacity zero. These results can