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On a Lemma of Tate-Thompson

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In his famous account [12], J. Tate stated that the algebraic cycles span all the (l-adic) cohomology groups of the hypersurface defined by the equation:

$$x_0^n + x_1^n + \cdots + x_r^n = 0$$

in the r-dimensional projective space $\mathbf{P}^r$ over an algebraically closed field k of characteristic p, if r is odd and $p^\nu \equiv -1 \bmod n$ for some ν. The statement can easily be reduced to the case that $n = q+1$ ($q = p^\nu$). The crucial point, which is due to Tate and Thompson, is that the middle-dimensional l-adic cohomology group $H^{r-1}(S, \mathbf{Q}_l)$ of the hypersurface S defined by the equation:

$$x_0^{q+1} + x_1^{q+1} + \cdots + x_r^{q+1} = 0 \quad \text{in} \quad \mathbf{P}^r,$$

breaks up into the sum of two irreducible $U_{r+1}(\mathbf{F}_q)$-modules, one of which is the trivial one, where $U_{r+1}(\mathbf{F}_q)$ is the finite unitary group of rank $r+1$ over the finite field $\mathbf{F}_q$ with q elements and $H^*(S, \mathbf{Q}_l)$ has the $U_{r+1}(\mathbf{F}_q)$-module structure given by the natural action of $U_{r+1}(\mathbf{F}_q)$ on S.

In this paper, we shall first, in § 1, give the identification of this non-trivial irreducible piece in $H^{r-1}(S, \mathbf{Q}_l)$ with a certain unipotent representation of $U_{r+1}(\mathbf{F}_q)$ classified by Lusztig-Srinivasan [10]. This argument also gives the proof of the above mentioned Tate-Thompson's statement. Secondly, in § 2, we shall determine the character of this irreducible representation, by a method similar to that of [9]. Since the arguments in § 2 are quite independent of those in § 1, one can immediately obtain an alternative proof of the irreducibility of the Tate-Thompson representation.

We understand that some parts of this paper, especially results in § 1, which are essentially easy exercises of Lusztig's results [8], may be known to experts. However, since Tate-Thompson's result just stated is Mecca of recent developments of the use of l-adic cohomologies in the representation theory of the finite linear groups, and since the original proof of Tate-Thompson does not seem to be highly available to many people, we consider it to be of some meaning that we write up the following account on these subjects. Of course, for various reasons from a historical point of view, one of our proofs of the irreducibility, given in § 1, seems to be different from that of Tate-Thompson.

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