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On a Functional of Distribution Functions having Maximum at Gaussian Distribution Function

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§ 1. Introduction

The entropy functional $H[f]$ is defined by

$$H[f] = -\int_{-\infty}^{\infty} f(x)\log f(x)\,dx, \qquad f \in \mathscr{D},$$

where $\mathscr{D}$ is the set of probability density functions f on R^1 with $\int f(x)|\log f(x)|dx < \infty$. Let $\mathscr{D}_1$ be the subset of $\mathscr{D}$ with $\int x^2 f(x)dx = 1$, and $g \in \mathscr{D}_1$ be the Gaussian density function with mean 0. Then Gibbs' lemma states that

$$H[f] \leqq H[g], \qquad f \in \mathscr{D}_1. \tag{1.1}$$

Consider a class of functionals $\tilde{H}[f]$ of the form

$$\tilde{H}[f] = \int_{-\infty}^{\infty} h(f(x))\,dx, \qquad f \in \mathscr{D}_1.$$

Under some regularity conditions on h, McKean[3] proved that if the inequality (1.1) holds with $H = \tilde{H}$, then $h(x) = c_1 x + c_2 x \log x$ $(c_2 \leqq 0)$.

Let $\mathscr{P}_1$ be the set of probability distribution functions with mean 0 and variance 1, and G be the Gaussian distribution function belonging to $\mathscr{P}_1$. Tanaka [6] considered the functional $\mathrm{e}[F]$ defined by

$$\mathrm{e}[F] = \inf \int_{R^2} |x-y|^2 dM(x, y), \qquad F \in \mathscr{P}_1,$$

where the infimum is taken over all two-dimensional distribution functions $M(x, y)$ whose marginals are F and G. It is known (see [6] or [4]) that

$$\mathrm{e}[F] = \int_0^1 |F^{-1}(\alpha) - G^{-1}(\alpha)|^2 d\alpha$$

$$= 2 - 2\Phi_0[F], \qquad \Phi_0[F] = \int_{-\infty}^{\infty} xG^{-1}(F(x))dF(x),$$

where $F^{-1}(\alpha)$ is the right continuous inverse function of $F(x)$. It can be proved