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An Extended Airy Function of the First Kind

Mitsuhiko KOHNO

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1. Introduction

The linear differential equation

$$(1.1)\qquad z^n \frac{d^n y}{dz^n} - z^q y = 0,$$

where z is a complex variable and q is an integer larger than n, has an extended form of the well-known Airy equation. For $n=2$ and $q=3$ (1.1) is exactly the Airy equation which has a long history of investigations. Two linearly independent entire solutions of the Airy equation $Ai(z)$ and $Bi(z)$ are called the Airy functions of the first and second kind, respectively. Their properties have been studied in great detail (see [5, 6]). For instance, we here give a brief exposition of the global behavior of the Airy function of the first kind

$$(1.2)\qquad Ai(z) = \sum_{m=0}^{\infty} \frac{z^{3m}}{3^{2m+2/3} m!\,\Gamma\left(m+\frac{2}{3}\right)} - \sum_{m=0}^{\infty} \frac{z^{3m+1}}{3^{2m+4/3} m!\,\Gamma\left(m+\frac{4}{3}\right)}.$$

$Ai(z)$ is recessive on the positive real axis $\arg z=0$ and admits the following asymptotic behavior as z tends to infinity:

$$(1.3)\qquad \begin{cases} Ai(z) \sim \dfrac{-i}{2\sqrt{\pi}} \exp\left(\dfrac{2}{3} z^{\frac{3}{2}}\right) z^{-\frac{1}{4}} \displaystyle\sum_{s=0}^{\infty} \left(\frac{3}{4}\right)^s \frac{\Gamma\left(s+\frac{1}{6}\right)\Gamma\left(s+\frac{5}{6}\right)}{\Gamma(s+1)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)} z^{-\frac{3}{2}s} \\ \qquad\qquad \text{in } -\dfrac{4}{3}\pi < \arg z < -\pi, \\ Ai(z) \sim \dfrac{1}{2\sqrt{\pi}} \exp\left(-\dfrac{2}{3} z^{\frac{3}{2}}\right) z^{-\frac{1}{4}} \displaystyle\sum_{s=0}^{\infty} \left(-\frac{3}{4}\right)^s \frac{\Gamma\left(s+\frac{1}{6}\right)\Gamma\left(s+\frac{5}{6}\right)}{\Gamma(s+1)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)} z^{-\frac{3}{2}s} \\ \qquad\qquad \text{in } -\pi < \arg z < \pi, \\ Ai(z) \sim \dfrac{i}{2\sqrt{\pi}} \exp\left(\dfrac{2}{3} z^{\frac{3}{2}}\right) z^{-\frac{1}{4}} \displaystyle\sum_{s=0}^{\infty} \left(\frac{3}{4}\right)^s \frac{\Gamma\left(s+\frac{1}{6}\right)\Gamma\left(s+\frac{5}{6}\right)}{\Gamma(s+1)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)} z^{-\frac{3}{2}s} \\ \qquad\qquad \text{in } \pi < \arg z < \dfrac{4}{3}\pi. \end{cases}$$