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Incomparability in Ring Extensions

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Introduction

Throughout this paper rings will be all commutative rings with units and morphisms will mean unitary ring-homomorphisms.

The purpose of this paper is to study some properties of *an incomparable morphism* (cf. [3]) and to introduce the notion of *universally incomparable morphisms* which will play an important role in this paper.

We shall discuss in § 1 some basic properties of an incomparable morphism. In § 2, we shall define a universally incomparable morphism and shall examine its properties. Let k be a field. For a k-algebra A, we shall prove in Theorem 2.9 that $k \to A$ is a universally incomparable morphism if and only if A is integral over k, and also if and only if $k[X] \to A[X]$ is an incomparable morphism. We shall also give in Theorem 2.11 and in Theorem 2.12 some necessary and sufficient conditions for a morphism f: $A \to B$ to be a universally incomparable one. Moreover, in Proposition 2.17, we shall show that if a morphism f of finite type is incomparable, then f is a universally incomparable morphism.

In § 3, we shall discuss incomparability for some special ring extensions. In Corollary 3.2, we shall give some necessary and sufficient conditions for a morphism: $A \to A[X]/I$ to be an incomparable one, where I is an ideal of $A[X]$. In Corollary 3.6, we shall also give two necessary and sufficient conditions for incomparability to hold for $A \to \bigotimes_{i=1}^{n} A[X]/I_i$, where I_i is an ideal of $A[X]$ for each i. In Proposition 3.11, we shall show that $A \to A[\alpha]$ is an incomparable morphism for each $\alpha \in \Omega$, where A is a Prüfer domain and Ω is the algebraic closure of the quotient field of A.

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