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Positive Solutions of Nonlinear Differential Inequalities

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1. Introduction

In this paper we are concerned with positive solutions of differential inequalities of the form

$$x^{(n)}(t) + p(t)f(x(g(t))) \leqq 0, \tag{I}$$

where $n \geqq 2$ and the following conditions are always assumed to hold:

(a) $p(t)$ is continuous, $p(t) \geqq 0$ for $t \geqq a$ and not eventually identically zero;

(b) $g(t)$ is continuously differentiable for $t \geqq a$ and

$$0 < \liminf_{t\to\infty} g'(t) \leqq \limsup_{t\to\infty} g'(t) < \infty;$$

(c) $f(x)$ is continuous and $f(x) > 0$ for $x > 0$.

In the oscillation theory of nonlinear differential equations one of the important problems is to find necessary and sufficient conditions for the equations under consideration to be oscillatory. With regard to the equation

$$x^{(n)}(t) + p(t)f(x(g(t))) = 0, \tag{E}$$

necessary and sufficient conditions in order that every solution of (E) be oscillatory have been established by restricting the nonlinearity $f(x)$ to various classes of functions. When $f(x) = x^\gamma$ (γ is a ratio of two positive odd integers and $\gamma \neq 1$), a characterization of oscillation of (E) was obtained in terms of an integral condition by Kiguradze [3] and Ličko and Švec [5]. See also Onose [8] and the references in [8] concerning further developments on this nonlinearity. When $f(x)$ is eventually nonincreasing, a characterization of oscillation of (E) was obtained by Koplatadze [4]. We refer to Mahfoud [6] for the related nonlinearity. On the other hand, when f is an arbitrary continuous function, under appropriate conditions on $p(t)$ necessary and sufficient conditions for oscillation were given by Burton and Grimmer [1, Theorem 9] and Mahfoud [7, Theorem 3]. It seems to us that little is known about the case f is arbitrary.

In this paper we investigate (I) in the latter direction to present some new results. Our main purpose is to characterize the existence of positive solutions of (I) without any restriction on $f(x)$. More precisely, when f is a general nonlinear function satisfying only (c), we give necessary and sufficient conditions for