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A note on Gruenberg algebras

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1. Let $\rho(L)$, $\mathfrak{e}(L)$ and $\bar{\mathfrak{e}}(L)$ denote respectively the Hirsch-Plotkin radical, the sets of left Engel and bounded left Engel elements of a Lie algebra L over a field $\mathfrak{k}$. The classes of abelian, nilpotent and solvable Lie algebras over $\mathfrak{k}$ are denoted respectively by $\mathfrak{A}$, $\mathfrak{N}$ and $\mathrm{E}\mathfrak{A}$. If $\mathfrak{X}$ is a class of Lie algebras, then $\mathrm{L}\mathfrak{X}$ and $\acute{\mathrm{E}}\mathfrak{X}$ denote respectively the classes of locally $\mathfrak{X}$-algebras and algebras with ascending $\mathfrak{X}$-series.

Simonjan [3] has shown that the class of Gruenberg algebras equals $\acute{\mathrm{E}}\mathfrak{A} \cap \mathrm{L}\mathfrak{N}$ over a field of characteristic 0. Amayo and Stewart have asked the following among "Some open questions" in [1]:

Question 40. *Over a field of characteristic* $p>0$, *suppose that* $L \in \acute{\mathrm{E}}\mathfrak{A} \cap \mathrm{L}\mathfrak{N}$. *Is it true that* $x \in L$ *implies* $\langle x \rangle$ asc L ?

In this note we shall give an affirmative answer to this question. This will be obtained as a collorary of the following theorem, which is proved over a field of characteristic 0 in [1, Theorem 16.4.2].

THEOREM 1. *Let L be a Lie algebra over a field $\mathfrak{k}$ of arbitrary characteristic.*
(a) *If* $L \in \acute{\mathrm{E}}\mathfrak{A}$, *then* $\rho(L) \subseteq \mathfrak{e}(L) = \{x \in L \mid \langle x \rangle \operatorname{asc} L\}$.
(b) *If* $L \in \mathrm{E}\mathfrak{A}$, *then* $\bar{\mathfrak{e}}(L) = \{x \in L \mid \langle x \rangle \operatorname{si} L\}$.

COROLLARY *Let L be a Lie algebra over a field $\mathfrak{k}$ of arbitrary characteristic belonging to* $\acute{\mathrm{E}}\mathfrak{A} \cap \mathrm{L}\mathfrak{N}$. *Then* $x \in L$ *implies* $\langle x \rangle \operatorname{asc} L$.

We employ notations and terminology in [1]. All Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified.

2. We show the following lemma on ascending series of a Lie algebra, which is an extension of Lemma 16 in [2].

LEMMA. *Let L be a Lie algebra and $x \in \mathfrak{e}(L)$. Assume that L has an ascending $\mathfrak{X}$-series where* $\mathfrak{X} = \mathfrak{A}$, $\mathrm{L}\mathfrak{N}$ *or* $\mathrm{LE}\mathfrak{A}$. *Then L has an ascending $\mathfrak{X}$-series with terms idealized by x.*

PROOF. Let $(L_\alpha)_{\alpha \leq \lambda}$ be an ascending $\mathfrak{X}$-series of L with an ordinal λ. Let H_α be the sum of $\langle x \rangle$-invariant subspaces of L_α ($\alpha \leqq \lambda$). Then H_α is the largest $\langle x \rangle$-invariant subalgebra of L_α (cf. [2, Lemma 15]). Clearly $H_0 = L_0 = 0$,