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Weakly ascendant subalgebras of Lie algebras

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Introduction

Maruo [4] introduced the notion of weak ideals generalizing that of subideals to study some kind of coalescence in Lie algebras. Recently Kawamoto [3] has considered N_k-pairs ($k \in \mathbf{N}$) and N_∞-pairs of subalgebras to study criteria for subideality and ascendancy in Lie algebras. For a subalgebra H of a Lie algebra L, the fact that (H, L) is an N_k-pair means that H is a k-step weak ideal of L. In this paper we shall introduce the notion of weakly ascendant subalgebras of a Lie algebra generalizing those of weak ideals and N_∞-pairs and investigate their properties.

The main results are as follows. If L is a hyperabelian Lie algebra of length λ and H is a μ-step weakly ascendant subalgebra of L, then H is a $\mu\lambda$-step ascendant subalgebra of L (Theorem 1). Therefore a subalgebra of a hyperabelian Lie algebra is weakly ascendant if and only if it is ascendant (Theorem 2). Every finitely generated, weakly ascendant subalgebra of a Lie algebra is at most of ω-step (Theorem 4). For a subset S of a generalized solvable Lie algebra L such that $\langle S\rangle$ is finite-dimensional and nilpotent, S is a left Engel subset of L if and only if $\langle S\rangle$ is weakly ascendant and if and only if $\langle S\rangle$ is ascendant (Theorem 5). For subalgebras $H \leq K_i$ ($i = 1,\cdots, n$) of a finite-dimensional Lie algebra, H is weakly ascendant of finite step in $\langle K_1,\cdots, K_n\rangle$ if and only if so is it in each K_i (Theorem 7).

1.

Throughout the paper, let L be a not necessarily finite-dimensional Lie algebra over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified, and let λ and μ be arbitrary ordinals.

We write $H \leq L$ when H is a subalgebra of L and $H \lhd L$ when H is an ideal of L.

A subalgebra H of L is a λ-step ascendant subalgebra of L, denoted by $H \lhd^{\lambda} L$, provided there is a series $(H_\alpha)_{\alpha\leq\lambda}$ of subalgebras of L such that

(a) $H_0 = H$ and $H_\lambda = L$,

(b) $H_\alpha \lhd H_{\alpha+1}$ for any ordinal $\alpha < \lambda$,

(c) $H_\beta = \bigcup_{\alpha<\beta} H_\alpha$ for any limit ordinal $\beta \leq \lambda$.