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On null sets for extremal distances of order 2 and harmonic functions

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Introduction

L. Ahlfors and A. Beurling [1] gave a characterization of the removable singularities for the class of analytic functions with finite Dirichlet integral, in terms of extremal distances on the complex plane. In the N-dimensional euclidean space R^N, Väisälä [9] introduced the notion of null sets for extremal distances of order N, namely, NED-sets, and gave measure-theoretic conditions for NED-sets.

In the present paper, we are concerned with the extremal distances of order 2 in the N-dimensional space R^N with $N \geqq 3$ and give several characterizations of null sets for these extremal distances. More precisely, we shall consider the following three kinds of null sets. Given a compact set E in R^N, denote by $(\hat{E}^c)_Q$ the Kerékjártó-Stoïlow compactification of E^c $(=R^N-E)$ and by $(\hat{E}^c)_I$ the Aleksandrov compactification of E^c. Let B_0, B_1 be two disjoint closed balls in E^c, λ be the extremal distance of order 2 between B_0 and B_1 and λ_0 (resp. λ_Q, λ_I) be the extremal distance of order 2 between B_0 and B_1 relative to $E^c-(B_0 \cup B_1)$ (resp. $(\hat{E}^c)_Q-(B_0 \cup B_1)$, $(\hat{E}^c)_I-(B_0 \cup B_1)$). If $\lambda_0=\lambda$ (resp. $\lambda_Q=\lambda$, $\lambda_Q=\lambda_I$) for every choice of B_0 and B_1, then we call E an NED_2-set (resp. NED_2^Q-set, $NED_2^{Q,I}$-set).

Corresponding to these extremal distances, there are notions of 2-capacities of condenser, which were studied by many authors (for example, see [12], [5], [11]); and there are also notions of principal functions (see [8]). We shall give characterizations of NED_2-sets, NED_2^Q-sets and $NED_2^{Q,I}$-sets in terms of corresponding 2-capacities of condensers and principal functions.

Another characterizations will be given by the removability for certain classes of harmonic functions (cf. [5], [8], [11] for related results). Let G be a bounded domain containing E and let $HD^2(G)$ (resp. $HD^2(G-E)$) be the class of all harmonic functions with finite Dirichlet integrals on G (resp. on $G-E$). We shall say that E is removable for $\widetilde{HD}^2$ (resp. KD^2; $\widetilde{KD}^2$; HD^2) if every $u \in HD^2(G-E)$ with "vanishing normal derivative along E" (resp. with no flux; with no flux and "constant value along each component of E"; with no additional condition) can be extended to a function in $HD^2(G)$. (For precise definitions of $\widetilde{HD}^2$, KD^2, $\widetilde{KD}^2$, see § 3, as well as the references cited above.) It is well known (see [3])