HIROSHIMA MATH. J.
10 (1980), 381–384

Continuity of contractions in a functional Banach space

Yoshihiro MIZUTA

(Received December 25, 1979)

In the Dirichlet space theory, contractions on the real line play an important role in connection with potential theoretic properties. A. Ancona [1] proved that contractions are continuous in Dirichlet space. Our aim in this note is to prove that the contractions considered in [3] are continuous in a certain functional Banach space.

Let X be a locally compact space and ξ be a positive (Radon) measure on X. For measurable functions u and v on X, we define

$$u \vee v = \max\{u, v\}, \quad u \wedge v = \min\{u, v\},$$

$$u^+ = u \vee 0 \quad \text{and} \quad u^- = -(u \wedge 0).$$

Let $\mathscr{X} = \mathscr{X}(X; \xi)$ be a real reflexive Banach space whose elements are measurable functions on X. We denote by $\|u\|$ the norm of $u \in \mathscr{X}$, by $\mathscr{X}^*$ the dual space of $\mathscr{X}$, and by $\langle u^*, u\rangle$ the value of $u^* \in \mathscr{X}^*$ at $u \in \mathscr{X}$.

Throughout this note, let Φ be a strictly convex function on $\mathscr{X}$ such that

(i) $\Phi(u) \geqq 0$ for all $u \in \mathscr{X}$ and $\Phi(u) = 0$ if and only if $u = 0$;

(ii) if $\{u_n\} \subset \mathscr{X}$ and $\lim_{n\to\infty} \Phi(u_n) = 0$, then $u_n \to 0$ in $\mathscr{X}$;

(iii) Φ is bounded on each bounded subset of $\mathscr{X}$; and

(iv) Φ is differentiable in the sense of Gâteaux, i.e., there is an operator $G: \mathscr{X} \to \mathscr{X}^*$ such that for any $u, v \in \mathscr{X}$,

$$\langle Gu, v\rangle = \lim_{t\downarrow 0} \frac{\Phi(u+tv) - \Phi(u)}{t}.$$

The operator G is called the gradient of Φ and denoted by $\nabla\Phi$.

We shall use the following elementary properties of Φ and $\nabla\Phi$ without proof:

(Φ_1) Let $u \in \mathscr{X}$ and $u^* \in \mathscr{X}^*$. Then $u^* = \nabla\Phi(u)$ if and only if

$$\langle u^*, v-u\rangle \leqq \Phi(v) - \Phi(u) \qquad \text{for any} \quad v \in \mathscr{X}.$$

(Φ_2) $\nabla\Phi$ is bounded, i.e., it maps bounded sets in $\mathscr{X}$ to bounded sets in $\mathscr{X}^*$.

For a non-negative measurable function g on X, we define an operator T_g^+ by

$$T_g^+ u = u^+ \wedge g \qquad \text{for} \quad u \in \mathscr{X}.$$