HIROSHIMA MATH. J.
10 (1980), 375–379

Z-transforms and noetherian pairs

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(Received December 24, 1979)

Let A be a noetherian ring, and let Z be a subset of Spec(A) which is stable under specialization. Assume that every element of Z is a regular prime ideal. Let M be an A-module such that every A-regular element is M-regular. The Z-transform $T(Z, M)$ of M is a subset of $M\otimes_A Q(A)$ defined as follows:

$$T(Z, M) = \{x \in M \otimes_A Q(A) \mid V(M:_A x) \subseteq Z\},$$

where $Q(A)$ is the total quotient ring of A, $M:_A x=\{a\in A \mid ax\in M\}$, and $V(M:_A x)$ is the set of prime ideals of A containing $M:_A x$. Since $A:_A(x+y)$ and $A:_A xy$ contain $(A:_A x)(A:_A y)$ for every x and y in $Q(A)$, $T(Z, A)$ is a subring of $Q(A)$ which contains A. It is easy to see that $T(Z, M)$ is a $T(Z, A)$-module. Note that $T(Z, M)=\Gamma(X, \mathscr{H}^0_{X/Z}(\tilde{M}))$ where $X=\mathrm{Spec}(A)$ and $\tilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module associated to M (cf. [2], Chap. IV, (5.9)).

In this paper, we shall give necessary and sufficient conditions on A so that $(A, T(Z, A))$ is a noetherian pair. For noetherian rings R and S with $R\subseteq S$, we say that (R, S) is a noetherian pair if every ring T, $R\subseteq T\subseteq S$, is noetherian. If Z is the set of all regular maximal ideals of A, then $T(Z, A)$ is the global transform A^g of A introduced by Matijevic in [3]. He proved that (A, A^g) is a noetherian pair if A is reduced.

Let $B=A/I$ where I is an ideal of A. Assume that $\mathrm{Ass}_A(B)\subseteq \mathrm{Ass}_A(A)$. Let $Z'=\{\mathfrak{p}/I \mid \mathfrak{p}\in Z \text{ and } \mathfrak{p}\supseteq I\}$. Then it is clear that every element of Z' is a regular prime ideal of B and $T(Z, B)=T(Z', B)$. Moreover we have a natural ring homomorphism ϕ: $T(Z, A)\to T(Z, B)$ whose kernel is $T(Z, I)=T(Z, A)\cap IQ(A)$. It should be remarked that $\phi(x)z=xz$ for every $x\in T(Z, A)$ and $z\in T(Z, B)$. In the case that Z is the set of all regular maximal ideals of A, $T(Z, B)$ is not the global transform of B in general. However if every maximal ideal of A is regular, then $T(Z, B)=B^g$.

Our main result is the following

THEOREM. *Let A be a noetherian ring, and let Z be a subset of* Spec(A) *which is stable under specialization. Assume that every element of Z is a regular prime ideal. Then the following conditions on A are equivalent.*

(1) *$(A, T(Z, A))$ is a noetherian pair.*

(2) (a) *$T(Z, A/\mathfrak{p})$ is a finite $A/\mathfrak{p}$-module for every $\mathfrak{p}\in \mathrm{Ass}_A(A)$ such that $A_\mathfrak{p}$ is not reduced, and*