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Scalar curvatures of left invariant metrics on some Lie groups

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In [1], J. Milnor gave many facts concerning curvatures of left invariant metrics on Lie groups. About scalar curvatures, he showed that

(1) if a Lie group G is solvable, then every left invariant metric on G is either flat, or else has strictly negative scalar curvature, and

(2) every left invariant metric on $SL(2, \mathbf{R})$ has strictly negative scalar curvature.

And he conjectured that if the universal covering group of a Lie group G is homeomorphic to Euclidean space then the conclusion of (1) holds. In this note we shall show that this conjecture is affirmative, that is, we have the following

THEOREM. *Let G be a Lie group such that the universal covering group of G is homeomorphic to Euclidean space. Then every left invariant metric on G is either flat or else has strictly negative scalar curvature.*

Let G be a Lie group with a left invariant metric $\langle\ ,\ \rangle$ and H a closed normal subgroup. In this note, we always consider the left invariant metrics $\langle\ ,\ \rangle_H$ on H and $\langle\ ,\ \rangle_{G/H}$ on G/H obtained from the metric of G naturally, so that the natural embedding from H into G is an isometry and the natural projection π from G to G/H is a submersion. We denote the sectional curvatures of G, G/H and H by κ, κ_* and $\bar{\kappa}$, and the scalar curvatures by $\rho(G)$, $\rho(G/H)$ and $\rho(H)$ respectively.

LEMMA 1. *Let G be a Lie group whose universal covering group is homeomorphic to Euclidean space and $\mathfrak{g}$ its Lie algebra. If G is not solvable, then $\mathfrak{g}=\mathfrak{s}_1+\mathfrak{g}_0$ (direct sum) where $\mathfrak{s}_1$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$ and $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$ such that the connected simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}_0$ is homeomorphic to Euclidean space.*

PROOF. Let $\mathfrak{g}=\mathfrak{s}+\mathfrak{r}$ (direct sum) be a Levi decomposition, where $\mathfrak{s}$ is a semisimple Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{r}$ is the radical of $\mathfrak{g}$. By the assumption, $\mathfrak{s}\neq 0$ and the connected simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{s}$ is homeomorphic to Euclidean space. Because of the fact that a connected simply connected simple Lie group homeomorphic to Euclidean space is locally