HIROSHIMA MATH. J.
10 (1980), 311–322

Modularity in Lie algebras

Ralph K. AMAYO and Jochen SCHWARZ

(Received October 31, 1979)

A subalgebra M of a Lie algebra L is termed modular in L ($M \text{ m } L$) if M is a modular element in the lattice formed by the subalgebras of L, i.e., if

(*) $\langle M, U\rangle \cap V = \langle U, M \cap V\rangle$ for all $U, V \leqq L$ with $U \leqq V$ and

(**) $\langle M, U\rangle \cap V = \langle U \cap V, M\rangle$ for all $U, V \leqq L$ with $M \leqq V$ hold.

Simple examples for modular subalgebras of a Lie algebra L are the quasi-ideals of L—$Q \leqq L$ is called a quasi-ideal of L ($Q \text{ q } L$) if Q is permutable with every subspace R of L, i.e., if $[Q, R] \subseteq Q+R$ for all $R \subseteq L$ ([1], p. 28).

That the reverse implication is not true is shown by the Lie algebra L ($L=\langle e\rangle+\langle f\rangle+\langle g\rangle$) defined over a field containing no pair of elements α, β such that $\alpha^2+\beta^2=-1$, with the following multiplication: $[e,f]=g$, $[f,g]=e$, $[g,e]=f$. L is simple, and every one-dimensional subalgebra of L is maximal and modular in L, but not a quasi-ideal of L.

We prove the following (M_L denotes the core of M in L):

(i) A modular subalgebra M of a Lie algebra L permutable with a solvable subalgebra A of L is a quasi-ideal of $M+A$ — in particular M is a quasi-ideal of L if L is solvable.

(ii) A modular subalgebra M of a finite-dimensional Lie algebra L over any field of characteristic zero is either

a) an ideal of L; or

b) L/M_L is metabelian, every subalgebra of L/M_L is a quasi-ideal, M/M_L is one-dimensional and is spanned by an element which acts as the identity map on $([L, L]+M_L)/M_L$; and $L/([L, L]+M_L)$ is one-dimensional; or

c) M/M_L is two-dimensional and L/M_L is the three-dimensional split simple Lie algebra; or

d) M/M_L is a one-dimensional maximal subalgebra of L/M_L and L/M_L is a three-dimensional non-split simple Lie algebra.

1. Elementary properties of modular subalgebras

The properties 1.1–1.3 hold for modular elements in more general lattices; proofs can be found in [9], where the modular elements are called "Dedekind elements."

PROPOSITION 1.1. *Let M be modular in a Lie algebra L and let U be a*