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On the covering properties of certain exceptional sets in a half-space

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§1. Introduction

Let $D=\{x\in\mathbf{R}^p: x_1>0\}$ where $x=(x_1,\dots, x_p)$ and $p\geq 2$. We shall say that a set $E\subset\mathbf{R}^p$ has a covering $\{r_n, R_n\}$ if there exists a sequence of balls $\{B_n\}$ in $\mathbf{R}^p$ such that $E\subset\bigcup_{n=1}^{\infty} B_n$, where r_n is the radius of B_n, and R_n is the distance between the origin and the centre of B_n. On the other hand, we shall say that $E\subset D$ has a covering $\{t_n, r_n, R_n\}$ if there exists a sequence of balls $\{B_n\}$ with centres in D such that $E\subset\bigcup_{n=1}^{\infty} B_n$, where r_n and R_n are defined as above and where t_n is the distance between the centre of B_n and the Euclidean boundary of D, to be denoted by ∂D.

The motivation of this work stems from two classical questions which are concerned with the behaviour at ∞ of a suitably restricted subharmonic function u on D. If u is subharmonic on D and $y\in\partial D$, we define $u(y)=\limsup_{x\to y} u(x)$, where $x\in D$. If $u(y)\leq 0$ for all $y\in\partial D$, and if $\sup_{x\in D}(u(x)/x_1)<\infty$, then it is generally known that u can be uniquely decomposed as

$$u(x) = \alpha x_1 - G\mu(x) - w(x),$$

where α is a real number, $G\mu$ is the Green potential of a mass distribution μ on D, and w is a positive harmonic function on D which can be represented as

$$w(x) = \int_{\partial D} K(y, x)dv(y)$$

where K is the Poisson kernel on $\partial D\times D$ and v is a suitable mass distribution on ∂D.

The first question, to be designated by (I), is concerned with an analysis of $u(x)/x_1$ as $x\to\infty$, $x\in D$. By introducing the idea of a minimally thin set at ∞ with respect to D, J. Lelong-Ferrand ([12], pp. 134–143) presented a solution of (I) by showing that for any $\varepsilon>0$, there exists a set $E_\varepsilon\subset D$, minimally thin at ∞ in D, such that

$$|u(x)/x_1 - \alpha| < \varepsilon, \quad x\in D\setminus E_\varepsilon, \quad |x|\geq 1.$$

She also proved that her results are best possible in the sense that if $E\subset D$ is