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Z-transforms and overrings of a noetherian ring

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Introduction

Since Nagata had pointed out the importance of the notion of ideal transforms in relation to the 14-th problem of Hilbert, ideal transforms have been studied by many authors. The notion of Z-transforms of a ring A, Z being a subset of Spec(A) which is stable under specialization, is a generalized one of ideal transforms. We can use ideal or Z-transforms as a powerful tool to study overrings B of a noetherian ring A. This is done as follows. Take a suitable chain $Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_0 = \mathrm{Spec}(A)$ of subsets of Spec(A) and consider the overrings $T(Z_i, A) \cap B$ where $T(Z_i, A)$ is the Z_i-transform of A. Then by examining properties of $T(Z_i, A) \cap B$ inductively, we get the knowledge of properties of B. K. Yoshida, in [22], used this technic and showed some properties of overrings B are determined by local properties at prime ideals in $\mathrm{Ass}_A(B/A)$. But the essential point of this technic is that we can reduce a problem on B to a problem on $(A_\mathfrak{p})^g \cap B_\mathfrak{p}$, $\mathfrak{p} \in \mathrm{Ass}_A(B/A)$, where $(A_\mathfrak{p})^g$ is the global transform of $A_\mathfrak{p}$. This motivation follows from two facts: The first one is a characterization of $\mathrm{Ass}_A(B/A)$, i.e. $\mathrm{Ass}_A(B/A) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_\mathfrak{p} \subset (A_\mathfrak{p})^g \cap B_\mathfrak{p}\}$ (Theorem (2.5)). On the other hand, roughly speaking, the difference between $T(Z_i, A) \cap B$ and $T(Z_{i-1}, A) \cap B$ appears in prime ideals belonging to $Z_{i-1} - Z_i$, and if $Z_{i-1} - Z_i$ is discrete, then $(T(Z_i, A) \cap B)_\mathfrak{p} = A_\mathfrak{p}$ and $(T(Z_{i-1}, A) \cap B)_\mathfrak{p} = (A_\mathfrak{p})^g \cap B_\mathfrak{p}$ for every $\mathfrak{p} \in Z_{i-1} - Z_i$. This is the second fact which we wish to point out. In this paper we shall study overrings of a noetherian ring from the above point of view.

Section 1 consists of preliminary results on Z-transforms and global transforms almost all of which are already known (cf. [1], [6], [9], [12], [13], [14] and [15]). We shall frequently use these results in this paper. In section 2, we shall give basic relations between $\mathrm{Ass}_A(B/A)$ and Z-transforms. We remark here that we shall obtain whole results in this section, especially Corollary (2.12), without using completions and the theorem of Mori-Nagata. Corollary (2.12) is a modified form of Theorem (1.6) in [14], and using this corollary we shall give an alternative proof of the theorem of Mori-Nagata in appendix (see [17] for another proof of this theorem by means of global transforms).

In some cases we can prove some known facts in a unified way by means of Z-transforms. In fact, in section 3, we shall generalize J. Nishimura's results [15, (2.6), (3.1) and (3.2)] (see Theorem (3.1)), and in the last part of section 5