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Generalized Cohen-Macaulay modules

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Introduction

The main purpose of this paper is to establish a notion of Cohen-Macaulay modules over an arbitrary commutative ring which generalizes that of Cohen-Macaulay modules over a noetherian, commutative ring. A finite module over a noetherian ring is said to be a Cohen-Macaulay module if its depth is equal to its Krull dimension (cf. [6]). Adapting M. Hochster's approach to a theory of grade, D. G. Northcott set up the concept of polynomial grade of modules over a commutative ring in [7] which is a generalization of the notion of depth. The author showed in [8] a relation between the polynomial grade of a module and the valuative dimension of it which was defined by P. Jaffard in [5]. Namely, let A be a quasi-local ring and M a non-zero, finite A-module. Then the polynomial grade $\mathrm{Gr}(M)$ of M is equal to or less than the valuative dimension $\mathrm{Dim}\, M$ of M. This fact suggests to us giving a definition of a Cohen-Macaulay module over an arbitrary ring in terms of polynomial grade and valuative dimension.

However it seems that many nice properties of Cohen-Macaulay modules over a noetherian ring come from the following inequality: $\mathrm{depth}\, M \leqq \dim A/\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ in $\mathrm{Ass}(M)$, where M is a non-zero, finite module over a noetherian local ring A. In particular it follows from this fact that a noetherian, Cohen-Macaulay ring is universally catenarian. First the author has guessed that a generalization of this inequality could be obtained. However S. Itoh has recently pointed out to the author that it does not hold in general, i.e., we can find a non-zero, finite module M over a quasi-local ring A and an attached prime ideal $\mathfrak{p}$ of M such that $\mathrm{Gr}(M) > \mathrm{Dim}(A/\mathfrak{p})$ (see Appendix). Therefore if we would define a Cohen-Macaulay module M over an arbitrary ring A by $\mathrm{Gr}(M) = \mathrm{Dim}\, M$, many nice properties of the Cohen-Macaulay modules over a noetherian ring may not be accomplished. For this reason, adding the condition that the ring $A/\mathrm{Ann}(M)$ is catenarian to the above one, we may introduce the following definition: A non-zero, finite module M over a ring A is said to be a *Cohen-Macaulay module* if $\mathrm{Dim}(M_{\mathfrak{p}})$ is finite for all $\mathfrak{p} \in \mathrm{Supp}(M)$ and $\mathrm{Gr}(M_{\mathfrak{q}}) + \mathrm{Dim}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \mathrm{Dim}\, M_{\mathfrak{p}}$ for all pairs of prime ideals $\mathfrak{p}$, $\mathfrak{q}$ in $\mathrm{Supp}(M)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ (see (4.4)). This would be a natural generalization of the notion of Cohen-Macaulay modules over a noetherian ring.

In section 1 we give the terminology and the notations which we will use in