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Nonlinear differential systems with monotone solutions

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Consider the vector equation

$$\mathbf{x}' = -\mathbf{f}(t, \mathbf{x}) \tag{1}$$

where $\mathbf{f}$: $[a, \infty) \times R^n \to R^n$ is continuous and sufficiently regular so that

(i) solutions of (1) vary continuously with initial data assigned at $t=a$,

and

(ii) solutions of (1) can be continued until some components become unbounded.

We seek to show the existence of a "monotone solution" $\mathbf{x}(t)$ of (1) whose components satisfy $x_i(t)>0$ and $x_i'(t)<0$ in $[a, \infty)$ for $1 \leq i \leq n$.

The existence of such monotone solutions was first established by Hartman and Wintner [5] under explicit bounds on $\mathbf{f}(t, \mathbf{x})$ which assured that all solutions of (1) can be continued to $t=\infty$. This requirement was removed in [6] for the case where n is even by means of a corollary of Sperner's lemma and for general n by means of Ważewski retracts in [2] (see also [3] and [4], Chapter XIV, Problems 2.8 and 2.9, for alternate techniques). The purpose of this note is to show how a different form of Sperner's lemma leads to the more general results obtained by the theory of retracts and also to note the nonlinear form of criteria which assure that the "monotone solution" of the scalar equation $y^{(n)}-(-1)^n \cdot f(t, y)=0$ tends to zero.

Writing $\mathbf{x} \geq \mathbf{0}$ ($\mathbf{x} > \mathbf{0}$) in case all components of a vector $\mathbf{x}$ satisfy $x_i \geq 0$ ($x_i > 0$), we formulate the following hypotheses

(A) $\mathbf{f}(t, \mathbf{x}) > \mathbf{0}$ whenever $\mathbf{x} > \mathbf{0}$ and $\mathbf{f}(t, \mathbf{0}) \equiv \mathbf{0}$;

(B) $f_i(t, \mathbf{x}) > 0$ whenever $\mathbf{x} \geq \mathbf{0}$ and $x_i = 0$ and some $x_j > 0$, for $1 \leq i \leq n$ and $j \neq i$.

These hypotheses are essentially satisfied when a scalar n-th order equation

$$y^{(n)} - (-1)^n f(t, y) = 0 \tag{2}$$

satisfying

(C) $f(t, y) > 0$ whenever $y > 0$ and $f(t, 0) \equiv 0$,

is represented as a first order system (1) by the transformation $x_i = (-1)^{i-1} y^{(i-1)}$, $1 \leq i \leq n$.

Our basic result is the following: