HIROSHIMA MATH. J.
11 (1981), 143–172

Fourier transforms and elementary solutions of invariant differential operators on homogeneous vector bundles over compact homogeneous spaces

Masaaki EGUCHI and Keisaku KUMAHARA

(Received August 25, 1980)

§ 1. Introduction

Let G be a connected compact Lie group and K be a closed subgroup of G. Let τ and σ be finite-dimensional unitary representations of K. We denote by E_τ the homogeneous hermitian vector bundle associated with τ. Let $\mathscr{D}(E_\tau)$ be the space of all C^∞ sections of E_τ with usual topology and $\mathscr{D}'(E_\tau)$ be its dual space. A homogeneous differential operator of $\mathscr{D}(E_\sigma)$ to $\mathscr{D}(E_\tau)$ is a left invariant differential operator on $\mathscr{D}(G/K)$ when $\tau=\sigma=$the identity representation of K. Let $\mathscr{D}$ be $\mathscr{D}(G)$ or $\mathscr{D}(G/K)$. In [1] Cerezo and Rouvière have determined when an invariant differential operator on $\mathscr{D}$ has an elementary solution, by using the Fourier transforms of $\mathscr{D}$ and $\mathscr{D}'$. On the other hand, N. R. Wallach has defined the Fourier transform on E_τ and determined the images of $\mathscr{D}(E_\tau)$ and $\mathscr{D}'(E_\tau)$ in [2].

The main purpose of the present paper is to generalize the notion of elementary solutions to vector bundle case and to characterize homogeneous differential operators which have elementary solutions. For this purpose we adopt a different definition from [2] of the Fourier transform as a direct generalization of [1].

Let V_τ be the representation space of τ. Sections of E_τ can be identified with V_τ-valued functions f which satisfy $f(xk)=\tau(k^{-1})f(x)$ for all $x\in G$ and $k\in K$. We first define the Fourier transforms of vector valued functions in § 2. In § 4 we study the images of $\mathscr{D}(E_\tau)$ and $\mathscr{D}'(E_\tau)$ by the Fourier transform which is the restriction of the above Fourier transform. In § 3 and § 5 we characterize homogeneous differential operators which have elementary solutions.

In [2] Wallach has determined which homogeneous differential operator is globally hypoelliptic. In § 6 we show when a globally hypoelliptic operator has an elementary solution.

§ 2. Fourier transforms of vector valued functions

2.1. Let G be a connected compact Lie group and let dx be the normalized Haar measure on G so that the total measure is one. For a finite-dimensional