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Regular modules and V-modules

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Introduction and Notation. A ring R is called a (von Neumann) regular ring if for each a in R there exists an x in R such that $a=axa$. The notion of regularity has been extended to modules by D. Fieldhouse [6], R. Ware [20] and J. Zelmanowitz [21]. In this paper, following Zelmanowitz [21], we call a right R-module M *regular* if given any $m \in M$ there exists $f \in \mathrm{Hom}_R(M, R)$ with $mf(m)=m$. O. Villamayor has shown that every simple right R-module is injective if and only if every right ideal of R is an intersection of maximal right ideals. If a ring R satisfies these equivalent conditions, R is called a right V-ring. The notion of V-rings has been extended to modules by V. S. Ramamurthi [16] and H. Tominaga [19]. In this paper, following Tominaga [19], we call a module M_R a *V-module* if every R-submodule is an intersection of maximal R-submodules. Such a module M_R has also been called "co-semisimple" by K. R. Fuller [10]. The connections between the class of regular rings and the class of V-rings are studied by many authors (see the references of [7]).

In this paper, we shall consider the connections between the class of regular modules and the class of V-modules, and we shall study the relationship between these modules and their endomorphism rings. J. Fisher and R. Snider [9, Corollary 1.3] proved that a ring R is regular if and only if R is fully idempotent and every prime factor ring of R is regular. In §2, we shall extend this result to modules (Theorem 2.3). In §3, we consider V-modules and their endomorphism rings. We prove that a finitely generated projective module M_R is a V-module if and only if $\mathrm{End}_R(M)$ is a right V-ring and M_R is a self-generator. In §4, we prove that a module M_R over a *P.I.*-ring R is regular if and only if it is a locally projective V-module (Theorem 4.4). R. Ware [20, Proposition 2.5] proved that if a projective module M_R over a commutative ring R is regular, then every simple homomorphic image of M_R is injective. The converse assertion was proved by V. S. Ramamurthi [16, Theorem 4] and Z. Maoulaoui [14, Proposition 1]. We shall prove this result for general regular modules over commutative rings. Finally, in §5, we consider fixed subrings of automorphisms. We prove that if G is a finite group of automorphisms of a ring R such that $|G|^{-1} \in R$ and $J(R/I)=0$ for every G-invariant right ideal I of R, then the fixed subring R^G is a right V-ring.

Throughout this paper, R will denote an associative ring with identity and all modules considered are unitary right R-modules. Homomorphisms will be