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Discrete series for an affine symmetric space

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§ 1. Introduction

We introduce the four dimensional linear space $\boldsymbol{R}^4$ with the bilinear form

$$[x, y] = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$$

defined on it. Let H^3 (resp. H_I^3) be the set of all lines passing through the origin of $\boldsymbol{R}^4$ and lying inside (resp. outside) the cone whose equation is $[x, x] = x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$, that is, all lines whose points satisfy the inequality $[x, x] < 0$ (resp. $[x, x] > 0$). Then naturally they may be interpreted as open submanifolds of the three dimensional projective space $P^3(\boldsymbol{R})$, and moreover they are homogeneous spaces:

$$H^3 = SO(3, 1)/S(O(3) \times O(1)) \quad \text{and} \quad H_I^3 = SO(3, 1)/S(O(1) \times O(2, 1)).$$

H^3 and H_I^3 are called the Lobachevskian space and the imaginary Lobachevskian space respectively. As is well known, in each $SO(3, 1)$-invariant riemannian structure on H^3 (such a structure exists) the space H^3 is a riemannian symmetric space. However, the imaginary Lobachevskian space H_I^3 has not an $SO(3, 1)$-invariant riemannian structure. Let us now go on to discuss "affine symmetric structure" on the space H_I^3.

For this purpose we consider the involutive automorphism σ of $SO(3, 1)$ defined by σ: $g \to J({}^tg)^{-1}J$, where $J = \text{diag.}(-1, 1, 1, -1)$. Then a simple calculation shows that the isotropy subgroup $S(O(1) \times O(2, 1))$ is exactly the set of all fixed points of σ.

On the other hand a manifold M with an affine connection is called an affine symmetric space if each $p \in M$ is an isolated fixed point of an involutive affine transformation s_p of M, which is called the symmetry at p. It is well known that the group of affine transformations $A(M)$ of M is a Lie group (see, [12]). Let $G = A(M)$ and let H be the isotropy subgroup at $p \in M$. Then M can be identified with G/H and s_p induces an involutive automorphism σ: $g \to s_p \circ g \circ s_p$ of G such that $(H_\sigma)_0 \subset H \subset H_\sigma$, where H_σ denotes the subgroup of G consisting of fixed points of σ and $(H_\sigma)_0$ is the identity component of H_σ.

Conversely, let G be a Lie group with an involutive automorphism σ and let H be a closed subgroup such that $(H_\sigma)_0 \subset H \subset H_\sigma$. Then the coset space G/H carries a canonical affine connection. Furthermore the manifold G/H is an affine