HIROSHIMA MATH. J.
11 (1981), 9–20

On the initial value problem for the Navier-Stokes equations in L^p spaces

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(Received June 25, 1980)

1. Introduction

On a bounded domain D in R^n $(n \geq 3)$ with smooth boundary S we consider the initial value problem for the Navier-Stokes equation

$$
\text{(N.S)} \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u, \operatorname{grad})u + \operatorname{grad} q = f & \text{in} \quad D \times (0, T), \\ \operatorname{div} u = 0 & \text{in} \quad D \times (0, T), \\ u = 0 & \text{on} \quad S \times (0, T), \\ u(x, 0) = a(x) & \text{in} \quad D. \end{cases}
$$

Here $u = u(x, t) = (u_1(x, t), \dots, u_n(x, t))$, $q = q(x, t)$ and $f = f(x, t) = (f_1(x, t), \dots, f_n(x, t))$ are the velocity, the pressure and the given external force respectively, and $(u, \operatorname{grad}) = \sum_j u_j \partial/\partial x_j$. Our main concern is in the existence and uniqueness problem of strong solutions of (N.S) in the Banach space $(L^p(D))^n$, $n < p < \infty$. In treating this problem we employ the method of Kato and Fujita [2], [7] and transform the equation (N.S) to the following evolution equation in the Banach space X_p:

$$
\text{(I)} \qquad \frac{du}{dt} + Au + P(u, \operatorname{grad})u = Pf, \quad t > 0, \quad u(0) = a \in X_p.
$$

Here X_p is the closed subspace of $(L^p(D))^n$ consisting of all solenoidal vector fields on D whose normal components vanish on S, and $A = -P\Delta$ is the Stokes operator with P denoting the projection onto X_p. See [4] for the details. Kato and Fujita [2], [7] considered the equation (I) in X_2, $n = 3$, and proved the existence and uniqueness, generally local in time, of strong solutions for initial data in $D(A^{1/4})$ under a certain assumption on Pf. In this paper we shall show that the above restriction on the initial data can be removed by considering (I) in X_p, $n < p < \infty$. Further, we show that, as is done in [2], [7], the solution exists globally if the data are sufficiently small. What is basic for our discussion is the estimation of the nonlinear term $P(u, \operatorname{grad})u$ by the fractional powers of the Stokes operator, the existence of which is assured by the fact that the Stokes