HIROSHIMA MATH. J.
11 (1981), 1–4

Ascendancy in locally solvable, ideally finite Lie algebras

Ernest L. STITZINGER

(Received April 23, 1980)

In a recent work, N. Kawamoto has obtained conditions which are sufficient for a subalgebra A to be ascendant in a generalized solvable Lie algebra L. One such condition is that for each $a \in L$, there exists $k = k(a)$ such that $(a)\,\mathrm{ad}^k\, x \in A$ for all $x \in A$. The results are obtained when the scalars come from a field of characteristic 0, a condition which is shown to be necessary for certain of the results. It seems to be of interest to obtain similar results without restrictions on the characteristic. Such a result is shown here and some consequences are derived.

The Lie algebras considered here are assumed to be over a field. The algebras are assumed locally solvable and ideally finite (see [3]); that is, each element of the algebra is contained in a finite dimensional ideal. Let $\mathfrak{J}$ denote the class of locally solvable, ideally finite Lie algebras. Let $L \in \mathfrak{J}$ and A be a subalgebra of L. For each $a \in A$, let $L_0(a) = \{x \in L;\ (x)\,\mathrm{ad}^k\, a = 0$ for some $k = 1, 2, \dots\}$ and $L_1(a) = \cap_{k=1}^{\infty} \mathrm{range}\,(\mathrm{ad}^k\, a)$. Since $L \in \mathfrak{J}$, clearly $L = L_0(a) + L_1(a)$ for each $a \in L$. In the conclusion of the main result, a condition which is apparently stronger than ascendancy is obtained. A is ω-ascendant in L if there exists a chain $A = A(0) \lhd A(1) \lhd \cdots A(\omega) = L$ where $A(\omega) = \cup_{k=0}^{\infty} A(k)$. The conditions which are sufficient for ω-ascendancy are also necessary. This is the context of the following main result.

THEOREM 1. *Let $L \in \mathfrak{J}$ and let A be a subalgebra of L. Then the following are equivalent*:

1. *A is ω-ascendant in L.*
2. *$A + L_0(a) = L$ for all $a \in A$.*
3. *$L_1(a) \subseteq A$ for all $a \in A$.*

PROOF. That 1 implies 3 is clear. Assume that 3 holds. Let $a \in A$, $x \in L$ and B be a finite dimensional ideal which contains x. Then $B = B_0(a) + B_1(a)$ and $x \in A + B = A + B_0(a) \subseteq A + L_0(a)$. Hence $L = A + L_0(a)$ and 2 follows. Now assume that 2 holds. L is the union of finite dimensional ideals $\{H(\lambda)\}$. Hence each $H(\lambda)$ contains a chain $0 = H(\lambda, 0) \subset \cdots \subset H(\lambda, n(\lambda)) = H(\lambda)$ where each $H(\lambda, i)$ is an ideal in L and $H(\lambda, i)/H(\lambda, i-1)$ is an irreducible L-module. Since L is locally solvable and $H(\lambda, 1)$, $H(\beta, 1)$ are minimal, $H(\lambda, 1)H(\beta, 1) = 0$. Consider $H(\lambda, j)$, $H(\beta, k)$. Then $T = H(\lambda, j-1) + H(\beta, k-1)$ is an ideal in L and