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On pseudo-Runge-Kutta methods of the third kind

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1. Introduction

Consider the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{1.1}$$

where the function $f(x, y)$ is assumed to be sufficiently smooth. Let $y(x)$ be the solution of this problem,

$$x_n = x_0 + nh \qquad (n = 1, 2,\dots, h > 0), \tag{1.2}$$

and let y_1 be an approximation of $y(x_1)$ obtained by some appropriate method. We are concerned with the case where a pseudo-Runge-Kutta method is used for computing approximations y_j of $y(x_j)$ $(j=2, 3,\dots)$.

Byrne and Lambert [1] introduced pseudo-Runge-Kutta methods of order $r+1$ $(r=2, 3)$ of the form

$$y_{n+1} = y_n + h \textstyle\sum_{i=1}^{r} (p_i k_{i,n} + q_i k_{i,n-1}) \qquad (n = 1, 2,\dots), \tag{1.3}$$

where

$$k_{i,m} = f(x_m + a_i h, y_m + h \textstyle\sum_{j=1}^{i-1} b_{ij} k_{j,m}) \qquad (i = 1, 2,\dots, r;\ m = 0, 1,\dots),$$
$$a_1 = 0,\ a_i = \textstyle\sum_{j=1}^{i-1} b_{ij} \qquad (i = 2, 3,\dots, r),$$

and p_i, q_i $(i=1, 2,\dots, r)$ and b_{ij} $(i=2, 3,\dots, r; j=1, 2,\dots, i-1)$ are constants. Gruttke [3] has shown that such a method exists also for $r=4$.

Costabile [2] considered pseudo-Runge-Kutta methods of the second kind of the form

$$y_{n+1} = y_n + h \textstyle\sum_{i=0}^{r} p_i k_i \qquad (n = 1, 2,\dots), \tag{1.4}$$

where

$$k_0 = f(x_{n-1}, y_{n-1}), \quad k_1 = f(x_n, y_n),$$
$$k_i = f(x_n + a_i h, y_n + h \textstyle\sum_{j=0}^{i-1} b_{ij} k_j), \quad a_i = \textstyle\sum_{j=0}^{i-1} b_{ij} \qquad (i = 2, 3,\dots, r),$$

and p_i $(i=0, 1,\dots, r)$ and b_{ij} $(i=2, 3,\dots, r; j=0, 1,\dots, i-1)$ are constants. Nakashima [4] proposed those of the third kind which are of the form (1.4) with