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Semi-fine limits and semi-fine differentiability of Riesz potentials of functions in L^p

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1. Statement of results

In the n-dimensional Euclidean space R^n, we define the Riesz potential of order α, $0<\alpha<n$, of a non-negative measurable function f on R^n by

$$U_\alpha^f(x) = R_\alpha * f(x) = \int |x-y|^{\alpha-n} f(y)dy; \qquad R_\alpha(x) = |x|^{\alpha-n}.$$

For a set E in R^n and an open set G in R^n, we set

$$C_{\alpha,p}(E;\, G) = \inf \|f\|_p^p,$$

where $\|f\|_p$ denotes the L^p-norm in R^n, $1<p<\infty$, and the infimum is taken over all non-negative measurable functions f on R^n such that $f=0$ outside G and $U_\alpha^f(x) \geqq 1$ for every $x \in E$.

A set E in R^n is said to be (α, p)-semi-thin at $x^0 \in R^n$ if

$$\lim_{r\downarrow 0} r^{\alpha p-n} C_{\alpha,p}(E \cap B(x^0, r) - B(x^0, r/2);\, B(x^0, 2r)) = 0,$$

where $B(x^0, r)$ denotes the open ball with center at x^0 and radius r. We note here that E is (α, p)-semi-thin at x^0 if and only if

$$\lim_{i\to\infty} 2^{i(n-\alpha p)} C_{\alpha,p}(E_i;\, G_i) = 0,$$

where $E_i = \{x \in E;\, 2^{-i} \leqq |x-x^0| < 2^{-i+1}\}$ and $G_i = \{x \in R^n;\, 2^{-i-1} < |x-x^0| < 2^{-i+2}\}$.

THEOREM 1 (cf. [2; Theorem 2]). *Let $0<\beta<(n-\alpha p)/p$, and f be a non-negative measurable function on R^n such that $U_\alpha^f \not\equiv \infty$. If*

$$\lim_{r\downarrow 0} r^{(\alpha+\beta)p-n} \int_{B(x^0,r)} f(y)^p dy = 0, \tag{1}$$

then there exists a set E in R^n such that E is (α, p)-semi-thin at x^0 and

$$\lim_{x\to x^0, x\in R^n-E} |x-x^0|^\beta U_\alpha^f(x) = 0.$$

REMARK 1. (i) (cf. [2; Theorem 2]) If $\alpha p = n$ and f is a non-negative measurable function in $L^p(R^n)$ such that $U_\alpha^f \not\equiv \infty$, then there exists a set E in R^n with the following properties: