HIROSHIMA MATH. J.
11 (1981), 493–498

Subideals and serial subalgebras of Lie algebras

Ian STEWART

(Received March 24, 1981)

The purpose of this note is to record several facts about the interplay between finiteness conditions on a Lie algebra, and the structure of its subideals and serial subalgebras (in the sense of [2] pp. 9, 258). Wielandt [16] has shown that a subgroup H of a finite group G is subnormal in G if and only if H is subnormal in $\langle H, g\rangle$ for every $g \in G$. This, and related criteria given by Wielandt in the same paper, have been extended to various classes of infinite groups by Hartley and Peng [7] and Whitehead [14, 15]. We obtain similar results for various classes of Lie algebras, though with somewhat different proofs owing to the unavailability of conjugacy arguments. In particular we prove an analogue of Wielandt's theorem for finite-dimensional Lie algebras over a field of characteristic zero. Chao and Stitzinger [3] prove a similar result for finite-dimensional soluble Lie algebras in arbitrary characteristic: their proof can be greatly simplified, and we do this in Theorem 2.

A generalization to locally finite Lie algebras leads to a criterion for a subalgebra of a locally finite Lie algebra to be serial, implying that a simple locally finite Lie algebra cannot have non-trivial serial subalgebras. (The group-theoretic analogue of this result appears to be unknown.) This is reminiscent of a theorem of Levič [9] and Amayo [1] on the nonexistence of ascendant subalgebras in arbitrary simple Lie algebras.

Notation for Lie algebras will follow Amayo and Stewart [2]. In particular '$\leq$', '$\lhd$', 'si', 'asc', and 'ser' denote the relations 'subalgebra', 'ideal', 'subideal', 'ascendant subalgebra', 'serial subalgebra' respectively (see [2] pp. 9, 10, 258). Triangular brackets $\langle\ \rangle$ denote the subalgebra generated by their contents. If L is a Lie algebra the *Fitting radical* $\nu(L)$ is the sum of the nilpotent ideals of L (equal to the nil radical in finite dimensions) and the *Hirsch-Plotkin radical* $\rho(L)$ is the unique maximal locally nilpotent ideal. If L is finite-dimensional we write $\sigma(L)$ for the soluble radical. In characteristic zero both $\nu(L)$ and $\sigma(L)$ are characteristic ideals (that is, invariant under derivations, see Jacobson [8] p. 74 and [2] p. 116). If L has finite dimension and the ground field has characteristic zero then $\nu(L)$ contains every nilpotent subideal of L ([2] p. 114). If $x, y \in L$ we write

$$[x, {}_ny] = [x, y, y, \dots, y] \qquad (n \text{ repetitions of } y)$$

with similar notation for subspaces. We put