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An infinite-dimensional semisimple Lie algebra

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The following has been an open question which is asked at the end of [1]: If a Lie algebra L over a field of characteristic zero is locally finite and semisimple, is L necessarily locally finite-dimensional-and-semisimple? We will give a negative answer to this question and investigate an interesting property of the Lie algebra constructed for the above purpose. We should notice that the converse of this question is true.

First we consider a well known Lie algebra. Let V be a vector space of infinite dimension over a field of characteristic zero, S the set of all transformations of V considered as a Lie algebra, A the set of elements of S of trace zero (in the sense in §4 of [2]) and F the set of elements of S of finite rank. It is shown in [2] that A is infinite-dimensional and simple and that $C_S(A)=\{x\in S: [x, A]=\{0\}\}$ is the set of scalar multiplications. It is easy to see that $F^2=A$ and F is locally finite. Further the only ideals of F are $\{0\}$, A and F. Let $\sigma(F)$ be the locally solvable radical of F. Since $[\sigma(F), A]=\{0\}$ by the fact that A is not locally solvable, $\sigma(F)\subseteq C_S(A)\cap F=\{0\}$. Thus F is semisimple.

Next we construct an infinite-dimensional semisimple Lie algebra. Let $\mathfrak{k}$ be a field of characteristic zero and S_i be a Lie algebra over $\mathfrak{k}$ with basis $\{x_i, y_i, h_i\}$ and multiplication $[x_i, y_i]=h_i$, $[x_i, h_i]=2x_i$, $[y_i, h_i]=-2y_i$ for $i=1, 2,\dots$. Let z be a derivation of $\oplus_{i=1}^{\infty} S_i$ defined by $x_i\mapsto 2x_i$, $y_i\mapsto -2y_i$, $h_i\mapsto 0$ for $i=1, 2,\dots$. Consider the split extension $L=\oplus_{i=1}^{\infty} S_i\dotplus \mathfrak{k}z$.

Let $\sigma(L)$ be the locally solvable radical of L and take an element $w=\sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m} b_j y_j+\sum_{k=1}^{p} c_k h_k+dz$ (a_i, b_j, c_k, $d\in\mathfrak{k}$) of $\sigma(L)$. Since $[w, S_{n+m+p+1}]\subseteq \sigma(L)\cap S_{n+m+p+1}=\{0\}$, we have $d=0$. Therefore $w\in\sigma(L)\cap\oplus_{i=1}^{\infty} S_i=\{0\}$. This implies that L is semisimple.

THEOREM. *There are locally finite and semisimple Lie algebras over a field of characteristic zero which are not locally finite-dimensional-and-semisimple.*

PROOF. Let M be a locally finite-dimensional-and-semisimple Lie algebra over a field of characteristic zero. Then for each element x of M there is a finite-dimensional and semisimple subalgebra F_x of M containing x. Since $M=\sum_{x\in M} F_x$ and $M^2\supseteq\sum_{x\in M} F_x^2=\sum_{x\in M} F_x=M$, we must have $M=M^2$.

Let F and L be the Lie algebras given above. Then $F^2=A\neq F$ and $L^2=$