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Dirichlet finite solutions of Poisson equations on an infinite network

Dedicated to Professor Makoto Ohtsuka on his 60th birthday

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§ 1. Introduction

Let X be a countable set of nodes, Y be a countable set of arcs, K be the node-arc incidence function and r be a strictly positive function on Y. The quartet $N=\{X, Y, K, r\}$ is called an infinite network if the graph $\{X, Y, K\}$ is connected, locally finite and has no self loop. For notation and terminologies, we mainly follow [4] and [5].

Let $L(X)$ be the set of all real functions on X and $L^+(X)$ be the subset of $L(X)$ which consists of non-negative functions. For $u \in L(X)$, the Laplacian $\Delta u \in L(X)$ is defined by

$$\Delta u(x) = - \sum_{y\in Y} K(x, y)r(y)^{-1} \sum_{z\in X} K(z, y)u(z)$$

and the Dirichlet integral $D(u)$ of u is defined by

$$D(u) = \sum_{y\in Y} r(y)^{-1}[\sum_{x\in X} K(x, y)u(x)]^2.$$

Denote by $\boldsymbol{D}(N)$ the set of all $u \in L(X)$ such that $D(u) < \infty$.

For $h \in L(X)$, we denote by $\boldsymbol{P}_h\boldsymbol{D}(N)$ the set of all Dirichlet finite solutions u of the discrete Poisson equation $\Delta u = h$, i.e.

$$\boldsymbol{P}_h\boldsymbol{D}(N) = \{u \in \boldsymbol{D}(N);\ \Delta u = h\}.$$

We say that $h \in L(X)$ is distinguished if $h \neq 0$ and $\boldsymbol{P}_h\boldsymbol{D}(N) \neq \emptyset$. This notion was introduced by M. Nakai and L. Sario [2] in order to study the existence of Dirichlet finite non-harmonic biharmonic functions on Riemannian manifolds. Our main purpose of this paper is to obtain discrete analogues of results in [2] concerning conditions for a given h to be distinguished. We shall also show that the distinguishedness of a given $h \in L(X)$ is related to the existence of flows with a current source.

In §2 we recall some facts of discrete Green potentials which play important roles in our study. Our main results are given in §3. Note that Theorem 3.4 has no counterpart in [2]. Relations between distinguishedness and existence of flows are discussed in §4. Results in this section have no counterparts in [2]