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Finitely generated subalgebras of generalized solvable Lie algebras

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Introduction

Recently many authors considered several conditions under which a subalgebra of a Lie algebra is a subideal or an ascendant subalgebra. Such conditions have been also investigated for groups. Especially Peng [4] and Whitehead [5] presented some criteria for a finitely generated subgroup to be subnormal.

In this paper we shall give several conditions which ensure that a finitely generated subalgebra of a Lie algebra is a subideal or an ascendant subalgebra. The following is our main result: When L is a solvable Lie algebra of not necessarily finite dimension over a field of characteristic zero, any subalgebra H of L generated by $\{h_1,\dots, h_n\}$ is a subideal of L if and only if there exists an integer $m\geq 0$ such that $L(\mathrm{ad}\, h_i)^m\subseteq H$ for $1\leq i\leq n$ (Theorem 1(a)). Conditions for a finitely generated subalgebra to be an ascendant subalgebra are also given (Theorem 1(b) and Theorem 2).

1. Preliminaries

Throughout this paper L will denote a Lie algebra of not necessarily finite dimension over a field $\mathfrak{k}$ of characteristic zero. We shall follow [1] for notation and terminology. In particular, H si L, H asc L, and $H\lhd^{\omega}L$ mean respectively that H is a subideal, an ascendant subalgebra, and an ω-step ascendant subalgebra of L, where ω is the first infinite ordinal. Triangular brackets $\langle\ \ \rangle$ denote the subalgebra of L generated by elements inside them.

$\mathfrak{F}$, $\mathfrak{N}$, E$\mathfrak{A}$ denote respectively the classes of finite dimensional, nilpotent, and solvable Lie algebras. A Lie algebra L belongs to the class É$\mathfrak{A}$ if there is an ordinal λ and an ascending series $(L_\alpha)_{\alpha\leq\lambda}$ of L whose factors $L_{\alpha+1}/L_\alpha$ are abelian. If in addition each L_α is an ideal of L, then L belongs to the class É$(\lhd)\mathfrak{A}$.

For $x, y\in L$ and an integer $n\geq 0$, we write $[x, {}_n y]=x(\mathrm{ad}\, y)^n$. The similar notation is used for subspaces. A derivation d of L is nil if for any finite dimensional subspace M of L there is an integer $n=n(M)\geq 0$ such that $Md^n=0$. We denote by $[\mathrm{End}(V)]$ the Lie algebra of all linear endomorphisms of a vector space V over $\mathfrak{k}$.

We begin with the following