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Mean values and associated measures of superharmonic functions

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1. Introduction

Throughout this paper Ω will denote a non-empty open subset of the Euclidean space $\mathbf{R}^n$ ($n \geq 2$). For each point x of $\mathbf{R}^n$ and each positive number r, let $B(x, r)$ and $S(x, r)$ denote, respectively, the open ball and the sphere of centre x and radius r. We shall use v to denote a superharmonic function in Ω.

If the closure $\bar{B}(x, r)$ of $B(x, r)$ is contained in Ω, then $v(x) \geq \mathscr{M}(v, x, r)$, where $\mathscr{M}(v, x, r)$ is the spherical mean value of v given by

$$\mathscr{M}(v, x, r) = (s_n r^{n-1})^{-1} \int_{S(x,r)} v ds.$$

Here s denotes surface area measure on $S(x, r)$ and s_n is the surface area of the unit sphere in $\mathbf{R}^n$. It is well known that if $B(x, R) \subseteq \Omega$, then $\mathscr{M}(v, x, \cdot)$ is decreasing on $(0, R)$ and $\mathscr{M}(v, x, r) \to v(x)$ as $r \to 0+$.

The measure ν associated to v is a non-negative (Radon) measure in Ω such that

$$\int_\Omega \phi d\nu = -(p_n s_n)^{-1} \int_\Omega v(x) \Delta \phi(x) dx$$

for each infinitely differentiable function ϕ with compact support in Ω. Here Δ is the n-dimensional Laplacian operator and $p_n = \max\{1, n-2\}$.

We are concerned here with a comparison of the behaviour of $\mathscr{M}(u, x, r)/\mathscr{M}(v, x, r)$ and $\mu(\bar{B}(x, r))/\nu(\bar{B}(x, r))$ as $r \to 0+$, where u is a superharmonic function in Ω with associated measure μ and x is a point of Ω such that $v(x) = +\infty$. As applications, we shall obtain results which restrict the size of the set of points at which, for example,

$$\limsup_{r \to 0+} r^\alpha \mathscr{M}(u, x, r) > 0 \qquad (n \geq 3, 0 < \alpha \leq n-2)$$

and we shall improve some recent results of Kuran [6] on superharmonic and harmonic extensions.

For the latter application, we shall need to work, more generally, with the case where u is δ-superharmonic in an open subset ω of Ω. Recall that u is said to be δ-superharmonic in ω if there exist superharmonic functions u_1 and u_2 in