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Kaplansky's radical and Hilbert Theorem 90 II

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Let F be a pre-Hilbert field, $K=F(\sqrt{a})$ be a non-radical extension of F (i.e. $a\notin R(F)$ where $R(F)$ is Kaplansky's radical of F) and $N: K\to F$ be the norm map. In [2], we introduced topologies on the groups $\dot{F}/\dot{F}^2$ and $\dot{K}/\dot{K}^2$ so that the norm map N is continuous and $R(F)$ is closed. We showed there that $N^{-1}(R(F))=(\dot{F}\cdot R(K))^-$, where the bar means the topological closure of $\dot{F}\cdot R(K)$.

In this paper we discuss the case where $K=F(\sqrt{a})$ is a radical extension of a quasi-pythagorean field F. A field F is called quasi-pythagorean if $R(F)=D_F\langle 1, 1\rangle=\{x\in\dot{F};$ the form $\langle 1, 1\rangle$ represents $x\}$. The main purpose of this paper is to give some properties of a quasi-pythagorean field F and show that $N^{-1}(R(F))=\dot{F}\cdot R(K)$. In the last section of this paper, we shall give an example of a quasi-pythagorean field F with $\dim R(F)/\dot{F}^2=n$ for any natural number n and $\dim \dot{F}/R(F)=\infty$.

§1. Preliminaries

In this section, we state some basic facts on Scharlau's method of transfer. By a field F, we shall always mean a field of characteristic different from two. Let $\dot{F}$ denote the multiplicative group of F. For a quadratic form φ_F over F, we define $D_F(\varphi)=\{a\in\dot{F};\ \varphi_F$ represents $a\}$ and $G_F(\varphi)=\{a\in F;\ a\varphi\simeq\varphi\}$. Let K be an extension field of F, and φ_F be a form over F. We denote $\varphi_F\otimes K$ by φ_K for simplicity.

Let $K=F(\sqrt{a})$ be a quadratic extension of F and $x=b+c\sqrt{a}$ $(b, c\in F)$ be an element of K. We write $Im(x)=c$ and $\bar{x}=b-c\sqrt{a}$. For any element $y\in\dot{K}$, we define the map s_y: $K\to F$ with $s_y(x)=Im(y\bar{x})$. It is clear that s_y is a non-zero F-linear functional, and for any non-zero functional s: $K\to F$, there exists a unique element $y\in\dot{K}$ such that $s=s_y$. For a form q_K over K, we denote the transfer of q_K with respect to s_y by $s_y^*(q_K)$.

LEMMA 1.1. *Let $K=F(\sqrt{a})$ be a quadratic extension of F. For $y\in\dot{K}$ and a form q_K over K, the following statements are equivalent:*
(1) *$s_y^*(q_K)$ is isotropic.*
(2) *$D_K(q_K)\cap y\dot{F}\neq\phi$.*

PROOF. We first assume that $s_y^*(q_K)$ is isotopic. Then there exists $x\in D_K(q_K)$