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A stochastic method for solving quasilinear parabolic equations and its application to an ecological model

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Introduction

We are concerned with the following Cauchy problem for a quasilinear parabolic equation:

$$(1.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\triangle u + \sum_{i=1}^{n} b_i(t, x; u)\dfrac{\partial u}{\partial x_i} + c(t, x; u)u, \quad t > 0,\ x \in \boldsymbol{R}^n, \\ u(0, x) = f(x) \geqq 0, \end{cases}$$

where $b_i(t, x; \cdot)$, $1 \leqq i \leqq n$, and $c(t, x; \cdot)$ are mappings defined for some functions $u: [0, \infty) \times \boldsymbol{R}^n \to \boldsymbol{R}$. We assume that the coefficients $b_i(t, x; u)$, $1 \leqq i \leqq n$, and $c(t, x; u)$ are independent of the future $\{u(s, y): s > t, y \in \boldsymbol{R}^n\}$ for each t. (See §1 for precise definition.)

Our main results are stated in §1 and §2. They are summarized as follows. The equation (1.1) has a unique solution which has a nice probabilistic expression (1.2) based upon an n-dimensional Brownian motion $\{B_t = (B_t^1, \dots, B_t^n), t \geqq 0\}$:

$$(1.2)\quad u(t, x) = E_x[f(B_t) \exp\left\{\int_0^t c(t-s, B_s; u)ds\right\} M_t(u)],$$

$$M_t(u) = \exp\left\{\sum_{i=1}^{n} \int_0^t b_i(t-s, B_s; u)dB_s^i - \frac{1}{2}\sum_{i=1}^{n} \int_0^t b_i(t-s, B_s; u)^2 ds\right\},$$

under some suitable conditions. In a special case where $b_i(t, x; u) = b_i(t, x, u(t, x))$, $1 \leqq i \leqq n$, and $c(t, x; u) = c(t, x, u(t, x))$, Freidlin [2] solved the Cauchy problem (1.1) by finding the unique solution of (1.2). Our results can be regarded as a generalization of Freidlin's. In §3, our theorem is applied to the equation

$$(3.1)\quad \frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}\left[\left(\int_x^{x+r} v(t, y)dy - \int_{x-r}^{x} v(t, y)dy\right)v\right], \quad t > 0,\ x \in \boldsymbol{R},$$

which appears in an ecological model. It can be proved that there exists a unique solution of (3.1) for each r, which is bounded for $0 \leqq t < \infty$ and continuous in the parameter $r \in [0, \infty]$. Here the expression (1.2) of the solution plays an essential role. We make two remarks on some related problems in §4; the one is on time-lag systems and the other is on Neumann problems.