

LCM-stableness in ring extensions

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Introduction

In his paper [4], R. Gilmer introduced the concept of LCM-stableness, relating to GCD-properties of a commutative group ring. The main purpose of this paper is to point out that, in some cases, the necessary and sufficient conditions for a ring extension to be LCM-stable can be given in terms of polynomial grade, originally due to M. Hochster and developed by D. G. Northcott. For this purpose, we shall introduce two further notions, R_2 -stableness and G_2 -stableness, and investigate the relationship between LCM-stableness and them. In these discussions it is important to know when ' $\text{Gr}(I) \geq 2$ ' implies ' $\text{gr}(I) \geq 2$ '. We shall give in the last section an example of a finitely generated ideal I in an integral domain, with $\text{gr}(I) = 1$ and $\text{Gr}(I) \geq 2$.

In §2, we shall show that flatness, INC and LCM-stableness are all equivalent notions for a simple extension which satisfies some conditions (cf. Th. 2.7). In §3, we shall examine a relation between R_2 -stableness and G_2 -stableness, and study universality of LCM-stableness; namely, in Th. 3.5 we shall prove that $A \subset B$ is G_2 -stable if and only if $A[X] \subset B[X]$ is G_2 -stable, and also if and only if $A[X] \subset B[X]$ is R_2 -stable. As a corollary to this theorem, we can see that, in case A is locally a GCD-domain, $A \subset B$ is LCM-stable if and only if so is $A[X] \subset B[X]$.

In §4, we shall examine LCM-stableness of a simple extension $A \subset A[\alpha]$. Let I be the kernel of the canonical homomorphism of $A[X]$ onto $A[\alpha]$. We shall first show in Th. 4.3 that if $I = (f(X))$ ($f(X) \in A[X]$), then $A[Y] \subset A[\alpha][Y]$ is R_2 -stable if and only if $\text{Gr}(c(f)) \geq 3$. Moreover, we shall show in Th. 4.5 that, under some conditions, $A \subset A[\alpha]$ is R_2 -stable if and only if $\text{Gr}(c(f)) \geq 3$. In particular, we can show that if A is locally a GCD-domain, then $A \subset A[\alpha]$ is LCM-stable if and only if $\text{Gr}(c(I)) \geq 3$ (cf. Cor. 4.6).

In §5 and §6, we shall deal with the case of doubly generated extension $A \subset A[\alpha, \beta]$. In §5, we shall study a special case (cf. Th. 5.5). In §6, we shall consider the case where $K(\alpha), K(\beta)$ are linearly disjoint over the quotient field K of A . Firstly we shall treat the case when $A \subset A[\alpha]$ is (faithfully) flat (cf. Prop. 6.1, Th. 6.4), and secondly we shall examine the kernel $K_{\alpha, \beta}$ of the canonical homomorphism of $A[X, Y]$ onto $A[\alpha, \beta]$ by means of polynomial grade (cf. Prop. 6.6, Cor. 6.7, Prop. 6.8). Moreover, in case A is locally a GCD-domain, we shall give a characterization of LCM-stableness of $A \subset A[\alpha, \beta]$ (cf. Th. 6.10).