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# Exponential image and conjugacy classes in the group $O(3, 2)$

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## §1. Introduction

Let $G$ be a classical real linear Lie group, $\mathfrak{g}$ its Lie algebra and let $\exp: \mathfrak{g} \to G$ be the exponential map of $G$. It is now well known the description of conjugacy classes in $G$ and orbits in $\mathfrak{g}$ under the conjugation action of $G$, as seen in the paper [1] by N. Burgoyne and R. Cushman. In [2], D. Ž. Djoković has studied that which of conjugacy classes lies in the image of the exponential map, and he obtained the many results based on the conjugacy classes. It is of interest to determine which conjugacy classes lie in the interior, boundary or exterior of $\exp \mathfrak{g}$ in $G$, for the ordinary topology of $\mathfrak{g}$ and $G$. In this paper, we shall observe this for a special classical group.

In the papers [7] and [8], the author showed the following for $G = GL(n, R)$ or $G = SL(n, R)$: Let $x$ be an element in $G$. Then (i) $x$ is an interior point of $\exp \mathfrak{g}$ in $G$ if and only if $x$ has no negative eigenvalues, (ii) $x$ is a boundary point of $\exp \mathfrak{g}$ in $G$ if and only if $x$ has negative eigenvalues and the multiplicities of the negative eigenvalues are all even.

Let $O(p, q)$ be the orthogonal group of the signature $(p, q)$, $o(p, q)$ its Lie algebra and let $O_0(p, q)$ be the connected component of the identity element in $O(p, q)$. In the paper [9], for $p \geq q \geq 0$ the author showed that $\exp: o(p, q) \to O_0(p, q)$ is surjective if and only if $q = 0, 1$. Hence $O(2, 2)$ is the simplest one that $\exp: o(p, q) \to O_0(p, q)$ is not surjective.

In this paper, we give the complete table for $G = O(3, 2)$ that shows which of conjugacy classes lies in the interior, boundary or exterior of $\exp \mathfrak{g}$ in $G$, and we also give similar results on $O(2, 2)$ as a corollary. The main results are Theorem 9 and the corollaries in Section 4. In particular, the boundary of $\exp o(2, 2)$ in $O(2, 2)$ and the boundary of $\exp o(3, 2)$ in $O(3, 2)$ are characterized as follows:

(i) Let $x \in O(2, 2)$. Then $x$ is a boundary point of $\exp o(2, 2)$ in $O(2, 2)$ if and only if eigenvalues of $x$ are all real negative and the multiplicity of each eigenvalue of $x$ is even (2 or 4).

(ii) Let $x \in O(3, 2)$. Then $x$ is a boundary point of $\exp o(3, 2)$ in $O(3, 2)$ if and only if $x$ is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & x' \end{pmatrix}$ in $O(3, 2)$, where $x'$ is a boundary point of $\exp o(2, 2)$ in $O(2, 2)$.