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Quadratic extensions of quasi-pythagorean fields

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Let F be a field of characteristic different from 2 and K be a quadratic extension of F. We let $N: K \to F$ be the norm map and $R(F)$ (resp. $R(K)$) be Kaplansky's radical of F (resp. K). Formerly we proposed the following conjecture: Is $N^{-1}(R(F))$ equal to $\dot{F}\cdot R(K)$? In [3], we gave a necessary and sufficient condition under which both F and K are quasi-pythagorean (see §1) and showed that the conjecture is true in this case.

The purpose of this paper is to show that the conjecture is true, whenever F is quasi-pythagorean and satisfies the finiteness condition for the space of orderings (see Theorem 6.1).

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§1. Quasi-pythagorean fields

Throughout this paper, F shall be a field of characteristic not equal to 2. First we recall a few basic notation. For a field F, WF shall denote the Witt ring of F consisting of the Witt classes of all quadratic forms over F, and IF shall denote the fundamental ideal in WF consisting of the Witt classes of all even-dimensional forms. The notation $\langle a_1,\dots, a_n\rangle$ shall mean the diagonal form $a_1x_1^2+\cdots+a_nx_n^2$, where $a_i \in \dot{F}:=F\backslash\{0\}$. The nth power of the fundamental ideal shall be denoted by I^nF; it is additively generated by the n-fold Pfister forms $\langle\langle a_1,\dots, a_n\rangle\rangle:=\langle 1, a_1\rangle\otimes\cdots\otimes\langle 1, a_n\rangle$. For a form $f=\langle a_1,\dots, a_n\rangle$, we define $D_F(f)$ to be the set $\{\sum a_ix_i^2\neq 0; x_i \in F\}$. We note that if $n\geqq 2$, then $D_F\langle a_1,\dots, a_n\rangle = D_F\langle r_1a_1,\dots, r_na_n\rangle$ for $r_i \in R(F)$, where $R(F)$ is Kaplansky's radical of F. We also note that, for a Pfister form ρ and $x \in \dot{F}$, $x \in D_F(\rho)$ if and only if $\rho\otimes\langle\langle -x\rangle\rangle$ is isotropic.

As in [4], a field F is called quasi-pythagorean if $R(F)=D_F(2)$. It can be shown that F is quasi-pythagorean if and only if I^2F is torsion free. In [3], the subgroup H_a of F is defined by $H_a=\{x \in \dot{F}; D_F\langle 1, -x\rangle D_F\langle 1, -ax\rangle=\dot{F}\}$ and, in case F is quasi-pythagorean, it is shown that H_a is a subgroup of $D_F\langle 1, a\rangle$.

PROPOSITION 1.1. *Let F be a quasi-pythagorean field and $K=F(\sqrt{a})$ be a quadratic extension of F. Then the following statements are equivalent*:

(1) $N^{-1}(R(F))=\dot{F}\cdot R(K)$, *where N is the norm map $N: \to F$.*