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Convergence, consistency and stability of step-by-step methods for ordinary differential equations

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1. Introduction

Consider the initial value problem

$$y' = f(x, y) \quad (a \leqq x \leqq b), \quad y(x_0) = \eta, \tag{1.1}$$

where the function $f(x, y)$ is continuous and satisfies a Lipschitz condition with respect to y in $I \times R$, $I = [a, b]$, $R = (-\infty, \infty)$. Let $y(x)$ be the solution of this problem and let

$$x_n = a + nh \qquad (n = 0, 1, \dots, ; h > 0), \tag{1.2}$$

where h is a stepsize. We are concerned with the case where the approximations y_j $(j = 1, 2, \dots)$ of $y(x_j)$ are computed by step-by-step methods. Most of the conventional step-by-step methods such as one-step methods, linear multistep methods [1], hybrid methods [3], pseudo-Runge-Kutta methods [3, 4] and two-step methods [5] are of the form

$$\textstyle\sum_{j=0}^{k} a_j y_{n+j} = h\Phi(x_n, y_n, \dots, y_{n+k}; h) \qquad (n = 0, 1, \dots), \tag{1.3}$$

where a_j $(j = 0, 1, \dots, k)$ are real constants. Methods of this type determine y_{n+k} for given y_{n+i} $(i = 0, 1, \dots, k-1)$ and require starting values y_i $(i = 0, 1, \dots, k-1)$.

To achieve higher order with no increase in stepnumber, in this paper, besides the step nodes (1.2) we introduce $m-1$ sets of subsidiary nodes

$$x_{n+v_i} = a + (n+v_i)h \qquad (n = 0, 1, \dots; i = 2, 3, \dots, m) \tag{1.4}$$

and the approximations y_{in} of $y(x_{n+v_i})$, and consider the system of difference equations

$$\textstyle\sum_{i=1}^{m} \sum_{j=0}^{k_i} a_{ijq} y_{in+j} = h\Phi_q(x_n, y_{1n}, \dots, y_{1n+k_1}, \dots, y_{mn}, \dots, y_{mn+k_m}; h) \tag{1.5}$$
$$(n = 0, 1, \dots, N; q = 1, 2, \dots, m),$$

where $v_1 = 0$, $y_{1n} = y_n$ $(n = 0, 1, \dots)$, v_i $(i = 2, 3, \dots, m)$ are nonnegative numbers and a_{ijq} $(j = 0, 1, \dots, k_i;\ i, q = 1, 2, \dots, m;\ k_i \geqq 1)$ are real constants. Methods of this