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Kaplansky's radical and Hilbert Theorem 90 III

Tatsuo IWAKAMI, Daiji KIJIMA and Mieo NISHI

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Let F be a field, $R(F)$ be Kaplansky's radical of F and $K=F(\sqrt{a})$ be a quadratic extension of F. We showed in [4], that if F is a quasi-pythagorean field and K is a radical extension (i.e. $a \in R(F)-\dot{F}^2$), then K is also quasi-pythagorean and the '*H-conjecture*' $N^{-1}(R(F))=\dot{F}\cdot R(K)$ is valid, where $N: K \to F$ is the norm map.

In this paper we generalize the above results and show that the H-conjecture is valid whenever K is a quasi-pythagorean field.

§1. Preliminaries

Throughout the paper, let F be a field of characteristic different from two and $\dot{F}$ be the multiplicative group of F. We introduce in this section some subgroups of $\dot{F}$, and study their properties.

First, we put for $a \in \dot{F}$, $I_a=\{x \in \dot{F};\ D_F\langle 1, -a\rangle \subseteq D_F\langle 1, -x\rangle\}$.

PROPOSITION 1.1. *$I_a=\cap D_F\langle 1, -x\rangle$, where x runs over $D_F\langle 1, -a\rangle$. So I_a is a subgroup of $\dot{F}$.*

PROOF. If $b \in I_a$, then $D_F\langle 1, -a\rangle \subseteq D_F\langle 1, -b\rangle$ and we have $x \in D_F\langle 1, -b\rangle$ for all $x \in D_F\langle 1, -a\rangle$. Then $b \in D_F\langle 1, -x\rangle$ for all $x \in D_F\langle 1, -a\rangle$. So $b \in \cap D_F\langle 1, -x\rangle$, where x runs over $D_F\langle 1, -a\rangle$. Now, all the implications can be reversed and the proposition follows. Q.E.D.

PROPOSITION 1.2. *Let $K=(\sqrt{a})$ be a quadratic extension of F. Then the following statements hold:*

(1) $I_a=\{x \in \dot{F};\ \dot{F}\cdot D_K\langle 1, -x\rangle=\dot{K}\}$.
(2) $I_a \supseteq R(F)$, $I_a \ni a$.
(3) $I_a \supseteq R(K) \cap \dot{F}$.
(4) $R(K) \supseteq R(F)$.

PROOF. Let $N: K \to F$ be the norm map. Then $N(\dot{K})=D_F\langle 1, -a\rangle$ and, by the norm principle ([3], 2.13), we have $N^{-1}(D_F\langle 1, -x\rangle)=\dot{F}\cdot D_K\langle 1, -x\rangle$ for $x \in \dot{F}$. So, $D_F\langle 1, -a\rangle \subseteq D_F\langle 1, -x\rangle$ if and only if $\dot{K}=\dot{F}\cdot D_K\langle 1, -x\rangle$. This shows (1). The assertion (2) is clear and (3) follows from (1). The assertion (4)